LECTURE NOTES: STRICHARTZ ESTIMATES FOR THE WAVE EQUATION

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1. Introduction

We consider

$$\text{(IVP)} \qquad \begin{cases} u_{tt} - \Delta_x u = 0 \quad \text{in } \mathbb{R}^{n+1} \\ u(x,0) = g(x) \in \dot{L}^2_\alpha(\mathbb{R}^n) \\ u_t(x,0) = h(x) \in \dot{L}^2_{\alpha-1}(\mathbb{R}^n), \end{cases}$$

where the regularity parameter $\alpha \in \Big[0,(n+1)/(2n-2)\Big)$, and will be specified more precisely momentarily. We recall that for $\gamma \in [-1,1]$, we have that

$$\|f\|_{\dot{L}^2_\gamma(\mathbb{R}^n)} := \|(-\Delta)^{\gamma/2} f\|_2 = C_\gamma \,\| \, |\xi|^\gamma \hat{f}\|_2.$$

We may construct a solution to (IVP) via the Fourier transform formula

$$\hat{u}(\cdot,t)(\xi) = \frac{1}{2}\left(e^{it|\xi|} + e^{-it|\xi|}\right)\hat{g}(\xi) \, - \, \frac{i}{2}\left(e^{it|\xi|} - e^{-it|\xi|}\right)\frac{\hat{h}(\xi)}{|\xi|} =: \hat{u}_1 - \hat{u}_2, \tag{1.1}$$

as may be verified by a routine computation. We have the following

Theorem 1.2. *Let u be defined by (1.1), and suppose that the spatial dimension $n \geq 3$. We then have the "Strichartz estimate"*

$$\|u\|_{L^{q'}_t(L^{r'}_x)} \leq C_{q,r,n}\left(\|g\|_{\dot{L}^2_\alpha} + \|h\|_{\dot{L}^2_{\alpha-1}}\right), \tag{1.3}$$

where for $0 \leq \theta < 2/(n-1)$, we have

$$\alpha = \alpha(\theta) := \frac{\theta(n+1)}{4},$$

and $q' = q/(q-1)$, $r' = r/(r-1)$, with

$$r = r(\theta) := \frac{2}{1+\theta}, \quad q = q(\theta) := \frac{4}{4-\theta(n-1)}.$$

Remark. Notice that when $\theta = 2/(n+1)$, we have the case

$$\alpha = \frac{1}{2}, \quad r = q = p_n := \frac{2n+2}{n+3}.$$

Notice also that if $\theta = 0$, we have the case $\alpha = 0, r' = 2, q' = \infty$, which is an immediate consequence of (1.1) and Plancherel's theorem:

$$\sup_{t>0} \|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \|g\|_2 + \|\,|\xi|^{-1}\,\hat{h}\|_2 = \|g\|_2 + c\|h\|_{\dot{L}^2_{-1}}.$$

We need to let α vary with θ, owing to the interplay of homogeneities in different dimensions: $n+1$-dimensional scaling, but "n-dimensional" restriction (since the cone in $\mathbb{R}^{n+1}$ has only $n-1$ non-vanishing principal curvatures, as if it were a well-curved hypersurface in n dimensions).

The theorem is an easy consequence of a simple duality argument plus the "restriction theorem" of Stein-Tomas type for the cone:

Theorem 1.4. *Let θ, α, r and q be as above. We then have*

$$\left(\int_{\mathbb{R}^n} |\hat{f}(\xi, |\xi|)|^2 \frac{d\xi}{|\xi|^{2\alpha}}\right)^{1/2} \leq C_{q,r} \|f\|_{L^q_t(L^r_x)}.$$

We show now that Theorem 1.4 implies Theorem 1.2.

Proof. By duality, to obtain the conclusion of Theorem 1.2, it is enough to establish the bound

$$\left|\iint_{\mathbb{R}^{n+1}} u(x,t) f(x,t)\, dx dt\right| \leq C\|f\|_{L^q_t(L^r_x)} \left(\|g\|_{\dot{L}^2_\alpha} + \|h\|_{\dot{L}^2_{\alpha-1}}\right). \tag{1.5}$$

For the sake of notational simplicity, we ignore the factor of 1/2 in (1.1) and suppose either that $\hat{u} = e^{it|\xi|}\hat{g}$ or $\hat{u} = e^{it|\xi|}|\xi|^{-1}\hat{h}$, since the other pieces of u_1 and u_2 may be handled in the same way. The left side of (1.5) then equals either

$$C\left|\iint_{\mathbb{R}^{n+1}} f(x,t) \int_{\mathbb{R}^n} e^{i\xi\cdot x} e^{it|\xi|} \hat{g}(\xi) d\xi\, dx dt\right|, \tag{1.6}$$

or else the same expression with $|\xi|^{-1}\hat{h}(\xi)$ in place of $\hat{g}(\xi)$. In turn, (1.6) equals

$$\begin{aligned} \left|\int_{\mathbb{R}^n} \hat{g}(\xi) \iint_{\mathbb{R}^{n+1}} e^{i(x\cdot\xi + t|\xi|)} f(x,t)\, dx dt\, d\xi\right| &= \left|\int_{\mathbb{R}^n} \hat{g}(\xi) \check{f}(\xi, |\xi|) d\xi\right| \\ = \left|\int_{\mathbb{R}^n} |\xi|^\alpha\, \hat{g}(\xi) \check{f}(\xi, |\xi|)\, |\xi|^{-\alpha} d\xi\right| &\leq \|\, |\xi|^\alpha \hat{g}\|_2 \left(\int_{\mathbb{R}^n} |\check{f}(\xi, |\xi|)|^2 \frac{d\xi}{|\xi|^{2\alpha}}\right)^{1/2}. \end{aligned} \tag{1.7}$$

Similarly, replacing $\hat{g}$ by $|\xi|^{-1}\hat{h}(\xi)$ we obtain the bound

$$\|\, |\xi|^{\alpha-1}\hat{h}\|_2 \left(\int_{\mathbb{R}^n} |\check{f}(\xi, |\xi|)|^2 \frac{d\xi}{|\xi|^{2\alpha}}\right)^{1/2}.$$

In either case, since $\check{f} = \hat{\tilde{f}}$, where $\tilde{f}(x,t) = f(-x,-t)$, we have reduced matters to proving Theorem 1.4. ☐

The proof of Theorem will proceed in three steps:

(1) Prove Theorem 1.4 for a compact piece of the cone.
(2) Use a scaling argument to show that step 1 applies to all dyadic annuli on the cone.
(3) Use Littlewood-Paley theory to glue the dyadic pieces together.

We record now, for use in the sequel, the fundamental estimate used in Step 1. We recall that if σ denotes surface measure on a compact hypersurface $\mathbf{S} \subset \mathbb{R}^n$ with non-vanishing Gaussian curvature, then we have the decay estimate for the Fourier transform

$$|\widehat{\sigma}(\zeta)| \leq C|\zeta|^{-(n-1)/2}. \tag{1.8}$$

In particular, (1.8) holds when $\mathbf{S} = S^{n-1}$, the unit sphere in $\mathbb{R}^n$. This follows from the theorem proved in the course, it is the special case that $\mathbf{S}$ is compact, in which case we may take $\mu = \sigma$; another reference is [St2].

2. Step 1: Restriction theorem for a compact piece of the cone

In this section, we shall prove the following "compact" version of Theorem 1.4[1]:

$$\left(\int_{1/2\le|\xi|\le 2} |\hat{f}(\xi,|\xi|)|^2 \frac{d\xi}{|\xi|^{2\alpha}}\right)^{1/2} \le C_{q,r}\,\|f\|_{L^q_t(L^r_x)}. \tag{2.1}$$

To this end, let $\psi(|\xi|) = \tilde{\psi}(|\xi|)|\xi|^{-2\alpha}$, where $\tilde{\psi} \in C_0^\infty(1/4,4)$ and $\tilde{\psi} \equiv 1$ in $[1/2,2]$. Of course, we then have also that $\psi \in C_0^\infty(1/4,4)$. Moreover, the square of the left hand side of (2.1) is dominated by

$$\begin{aligned}\int |\hat{f}(\xi,|\xi|)|^2\psi(|\xi|)\,d\xi &= \int \hat{f}(\xi,|\xi|)\overline{\hat{f}(\xi,|\xi|)}\,\psi(|\xi|)\,d\xi \\ = \iint_{\mathbb{R}^{n+1}} f(x,t) \iint_{\mathbb{R}^{n+1}} &\left\{\int_{\mathbb{R}^n} e^{-i\xi\cdot(x-y)+|\xi|(t-s)}\psi(|\xi|)d\xi\right\} f(y,s)\,dyds\,dxdt \\ &=: . \iint_{\mathbb{R}^{n+1}} f(x,t) \iint_{\mathbb{R}^{n+1}} K_{t-s}(x-y)f(y,s)\,dyds\,dxdt,\end{aligned} \tag{2.2}$$

so it is enough to show that the operator

$$Vf(x,t) := \int_{-\infty}^{\infty} K_{t-s} * f(\cdot,s)\,ds = \iint_{\mathbb{R}^{n+1}} K_{t-s}(x-y)f(y,s)\,dyds$$

satisfies

$$\|Vf\|_{L^{q'}_t(L^{r'}_x)} \le C_{q,r}\|f\|_{L^q_t(L^r_x)}. \tag{2.3}$$

We observe that

$$K_{t-s}(x) = \left(e^{-i|\xi|(t-s)}\psi(|\xi|)\right)\hat{},$$

so by Plancherel's theorem we have

$$\sup_{t,s}\|K_{t-s} * g\|_2 \le C\|g\|_2 \tag{i}$$

We also claim that $|K_{t-s}(x)| \le C|t-s|^{-(n-1)/2}$, i.e., that

$$\|K_{t-s} * g\|_\infty \le C|t-s|^{-(n-1)/2}\|g\|_1 \tag{ii}$$

We defer the proof of the claim momentarily. Assuming that (ii) holds, we obtain by interpolation between (i) and (ii) that

$$\|K_{t-s} * g\|_{L^{r'}} \le C|t-s|^{-\beta}\|g\|_{L^r}, \tag{2.4}$$

where for $0 \le \theta < 2/(n-1)$, we have

$$\beta = \theta(n-1)/2$$

and

$$\frac{1}{r} = \frac{1-\theta}{2} + \frac{\theta}{1}.$$

By the Hardy-Littlewood-Sobolev theorem, we have that the fractional integral

$$I_{1-\beta}\,h(t) := \int_{-\infty}^{\infty} |t-s|^{-\beta}\,h(s)ds$$

satisfies, for $q = 4/(4-\theta(n-1)) = 2/(2-\beta)$,

$$\|I_{1-\beta}\,h\|_{L^{q'}(\mathbb{R})} \le C\|h\|_{L^q(\mathbb{R})}. \tag{2.5}$$

[1]This is really the main step.

Applying first (2.4) and then (2.5), we obtain (2.3):

$$\|Vf\|_{L_t^{q'}(L_x^{r'})} \leq C\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|t-s|^{-\beta}\|f(\cdot,s)\|_{L^r}ds\right)^{q'}dt\right)^{1/q'} \leq C\|f\|_{L_t^q(L_x^r)}.$$

It remains to prove the claim, i.e., the estimate (ii).

2.1. **Proof of (ii).** It is enough to show that

$$|K_t(x)| \leq C|t|^{-(n-1)/2}. \tag{2.6}$$

Recall that $K_t = \left(e^{-it|\xi|}\psi(|\xi|)\right)\hat{}$, so that

$$K_t(x) = C\int_{\mathbb{R}^n} e^{-ix\cdot\xi}e^{-it|\xi|}\psi(|\xi|)\,d\xi = C\int_{S^{n-1}}\int_0^{\infty} e^{-i(x\cdot\omega+t)\rho}\psi_0(\rho)d\rho d\sigma(\omega), \tag{2.7}$$

where we have used polar co-ordinates $\xi = \rho\omega$, $\rho = |\xi|$, $\omega = \xi/|\xi|$, and where $\psi_0(\rho) := \rho^{n-1}\psi(\rho) \in C_0^{\infty}(1/4, 4)$. In particular, we note that ψ_0 belongs to the Schwarz class $\mathcal{S}$; thus, so does its Fourier transform. We consider two cases:

Case 1. $|x| < |t|/100$.

In this case, since $|\omega| = 1$, we have that $|x\cdot\omega| << |t|$, so that $|x\cdot\omega + t| \approx |t|$. Moreover, the inner integral on the right hand side of (2.7) equals a constant times $\widehat{\psi_0}(x\cdot\omega + t)$. Since the latter is a Schwarz function, it follows that

$$|K_t(x)| \leq C_M\Big(1 + |x\cdot\omega + t|\Big)^{-M} \approx \Big(1 + |t|\Big)^{-M}, \quad \forall M < \infty.$$

Case 2. $|x| \geq |t|/100$.

In this case, we use Fubini's theorem to write

$$\begin{aligned}|K_t(x)| &= C\left|\int_0^{\infty} e^{-it\rho}\psi_0(\rho)\int_{S^{n-1}} e^{-i\rho x\cdot\omega}d\sigma(\omega)\,d\rho\right| \\ &= C\left|\int_0^{\infty} e^{-it\rho}\psi_0(\rho)\,\widehat{\sigma}(\rho x)\,d\rho\right| \leq C\int_0^{\infty}|\psi_0(\rho)\,|\rho x|^{-(n-1)/2}d\rho \\ &\leq C|t|^{-(n-1)/2},\end{aligned}$$

where in the second line we have used the decay estimate (1.8), and in the last line we have used that $1/4 < \rho < 4$ in the support of ψ_0, and that $|t| \leq C|x|$ in the present case. This concludes Step 1.

3. Step 2: the scaling argument

In this section, we use a scaling argument to extend (2.1) to an arbitrary dyadic annulus. To be precise, we shall prove

$$\sup_{k\in\mathbb{Z}}\left(\int_{2^{k-1}\leq|\xi|\leq 2^{k+1}}|\hat{f}(\xi,|\xi|)|^2\frac{d\xi}{|\xi|^{2\alpha}}\right)^{1/2} \leq C_{q,r}\,\|f\|_{L_t^q(L_x^r)}. \tag{3.1}$$

By the change of variable $\xi \to 2^k\xi$, we have

$$\begin{aligned}\int_{2^{k-1}\leq|\xi|\leq 2^{k+1}}|\hat{f}(\xi,|\xi|)|^2\frac{d\xi}{|\xi|^{2\alpha}} &= \int_{1/2\leq|\xi|\leq 2} 2^{k(n-2\alpha)}\left|\hat{f}\left(2^k(\xi,|\xi|)\right)\right|^2\frac{d\xi}{|\xi|^{2\alpha}} \\ &= \int_{1/2\leq|\xi|\leq 2}|\widehat{f_\alpha^k}(\xi,|\xi|)|^2\frac{d\xi}{|\xi|^{2\alpha}},\end{aligned} \tag{3.2}$$

where

$$\widehat{f^k_\alpha}(\zeta) := 2^{k(n-2\alpha)/2}\hat{f}(2^k\zeta);$$

i.e.,

$$f^k_\alpha(X) := 2^{-k(n+2+2\alpha)/2} f(2^{-k}X).$$

Applying (2.1) with f^k_α in place of f, we obtain from (3.2) that

$$\left(\int_{2^{k-1}\le|\xi|\le 2^{k+1}} |\hat{f}(\xi,|\xi|)|^2 \frac{d\xi}{|\xi|^{2\alpha}}\right)^{1/2} \le C_{q,r}\,\|f^k_\alpha\|_{L^q_t(L^r_x)}.$$

Consequently, to prove (3.1), it remains only to show that

$$\|f^k_\alpha\|_{L^q_t(L^r_x)} = \|f\|_{L^q_t(L^r_x)}, \tag{3.3}$$

whenever α, r, q are as in the statement of Theorem 1.2. To this end, we write the qth power of the left hand side of (3.3) as

$$\int_{-\infty}^{\infty}\left(\int_{\mathbb{R}^n}\left|2^{-k(n+2+2\alpha)/2} f\left(2^{-k}(x,t)\right)\right|^r dx\right)^{q/r} dt \tag{3.4}$$
$$= \int_{-\infty}^{\infty}\left(\int_{\mathbb{R}^n}|2^{-k(n+2+2\alpha)/2} f(x,t)|^r\, 2^{kn}dx\right)^{q/r} 2^k dt,$$

where we have made the change of variable $(x,t)\to 2^k(x,t)$. Therefore we need only show that

$$\frac{(n+2+2\alpha)r}{2} = n + \frac{r}{q}.$$

Plugging in the specified values of q, r, α, the latter is equivalent to showing that

$$\frac{n+2+\frac{\theta(n+1)}{2}}{1+\theta} = n + \frac{4-\theta(n-1)}{2(1+\theta)},$$

which in turn amounts to

$$2n+4+\theta(n+1) = 2n(1+\theta)+4-\theta(n-1).$$

At this point, the equality may be verified by trivial algebra.

4. Step 3: The Littlewood-Paley argument

Let $\eta\in C_0^\infty(1/2,2)$ satisfy the smooth partition of unity

$$\sum_{k=-\infty}^{\infty}\eta_k^2(\rho) := \sum_{k=-\infty}^{\infty}\eta^2(2^{-k}\rho) = 1,\ \ \rho\neq 0.$$

Set $\phi := (\eta(|\xi|))\check{}$, so that $\phi\in\mathcal{S}$, and $\int\phi = 0$. Let $\phi_k(x) := 2^{kn}\phi(2^k x)$, and define $Q_k g := \phi_k * g$, i.e.

$$\widehat{Q_k g}(\xi) = \eta_k(|\xi|)\hat{g}(\xi).$$

Thus, we have the discrete Calderón reproducing formula $\sum Q_k^2 = I$, and also the boundedness of the discrete Littlewood-Paley g-function:

$$\|\left(\sum|Q_k g|^2\right)^{1/2}\|_{L^r(\mathbb{R}^n)} \le C_r\|g\|_r,\ \ 1<r<\infty. \tag{4.1}$$

For a function f defined on $\mathbb{R}^{n+1}$, we may define $Q_k f$ via convolution in the space variable for each fixed t, i.e.,

$$Q_k f(x,t) := \int_{\mathbb{R}^n}\phi_k(x-y)f(y,t)\,dy.$$

We then have that

$$\int_{\mathbb{R}^n} |\hat{f}(\xi,|\xi|)|^2 \frac{d\xi}{|\xi|^{2\alpha}} = \sum \int_{\mathbb{R}^n} |\eta_k(|\xi|)\hat{f}(\xi,|\xi|)|^2 \frac{d\xi}{|\xi|^{2\alpha}} = \sum \int_{\mathbb{R}^n} |\widehat{Q_k f}(\xi,|\xi|)|^2 \frac{d\xi}{|\xi|^{2\alpha}}.$$

Since η_k is supported on $(2^{k-1}, 2^{k+1})$, we may apply (3.1) with $Q_k f$ in place of f to obtain

$$\int_{\mathbb{R}^n} |\hat{f}(\xi,|\xi|)|^2 \frac{d\xi}{|\xi|^{2\alpha}} \leq C_{q,r} \sum \|Q_k f\|^2_{L^q_t(L^r_x)}.$$

We now write the square root of the latter expresson as a constant times

$$\begin{aligned}
&\left(\sum \left(\int_{-\infty}^{\infty}\left(\int_{\mathbb{R}^n} |Q_k f(x,t)|^r \, dx\right)^{q/r} dt\right)^{2/q}\right)^{1/2} \\
&\qquad = \left\| \int_{-\infty}^{\infty}\left(\int_{\mathbb{R}^n} |Q_k f(x,t)|^r \, dx\right)^{q/r} dt \right\|_{\ell^{2/q}}^{1/q} \\
&\qquad \leq \left(\int_{-\infty}^{\infty} \left\| \left(\int_{\mathbb{R}^n} |Q_k f(x,t)|^r \, dx\right)^{q/r} \right\|_{\ell^{2/q}} dt\right)^{1/q} \\
&\qquad = \left(\int_{-\infty}^{\infty} \left\| \int_{\mathbb{R}^n} |Q_k f(x,t)|^r \, dx \right\|_{\ell^{2/r}}^{q/r} dt\right)^{1/q} \\
&\qquad \leq \left(\int_{-\infty}^{\infty} \left(\int_{\mathbb{R}^n} \|Q_k f(x,t)\|_{\ell^2}^r \, dx\right)^{q/r} dt\right)^{1/q} \leq C_{q,r} \|f\|_{L^q_t(L^r_x)},
\end{aligned}$$

where the third and fifth lines follow from Minkowski's integral inequality (since $q, r \leq 2$), and in the very last step we have used (4.1). This concludes the proof.

References

[St2] E. M. Stein *Harmonic Analysis*.