Convexity, Symmetrization, and Isoperimetry

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Remark: This is my (slightly more complete, but with far less foresight) notes for a class on convex geometry, as taught by Peter Pivovarov in University of Missouri.

1 Introduction

2 The Basics

Most of this section is taken from [1], Chapter 1. In particular, section 1.5-1.8.

2.1 The Spaces $\mathcal{C}^n$ and $\mathcal{K}^n$

Throughout this section, we will operate in $\mathbb{R}^n$ with the usual inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $\|x\|_2 = \sqrt{\langle x, x \rangle}$. Also denote B as the closed unit Euclidean ball.

Definition 2.1. Define the classes

$$\begin{aligned} \mathcal{C}^n &= \{K \subseteq \mathbb{R}^n \mid K \text{ non-empty and compact}\} \\ \mathcal{K}^n &= \{K \subseteq \mathbb{R}^n \mid K \text{ non-empty, compact, and convex}\} \end{aligned}$$

When we refer to a 'body' below, it'll be an element of either of these spaces.

It's not hard to see that $\mathcal{C}^n$ (and with a bit more effort, $\mathcal{K}^n$) is a vector space. More than that, if we can impose a sense of distance on the elements of these spaces, then they'll become metric spaces. Indeed, the appropriate metric is given by the Hausdorff distance.

Definition 2.2. For $K, L \in \mathcal{C}^n$, define the Hausdorff distance by

$$\delta^H(K, L) = \max \left\{ \sup_{x \in K} \inf_{y \in L} \|x - y\|_2 \, , \, \sup_{y \in L} \inf_{x \in K} \|x - y\|_2 \right\}$$

As this definition is somewhat opaque in its meaning, it is often convenient to rephrase the definition.

Lemma 2.3. *Given* $K, L \in \mathcal{C}^n$*,*

$$\delta^H(K, L) = \inf\{\delta \geq 0 \mid K \subseteq L + \delta B \text{ and } L \subseteq K + \delta B\}$$

Proof. □

Lemma 2.4. *δ^H is a metric on $\mathcal{C}^n$.*

Proof. $\square$

With this basics in place, now we can show that indeed δ^H is the 'correct' metric for the space $\mathcal{C}^n$.

Theorem 2.5. *$(\mathcal{C}^n, \delta^H)$ is a complete metric space.*

The proof will require the following continuity lemma.

Lemma 2.6. *Let $(K_i)_{i=1}^{\infty}$ be a decreasing sequence in $\mathcal{C}^n$. Then $\lim_{i\to\infty} K_i = \cap_j K_j$ in δ^H.*

Proof. We will use the alternate definition of Lemma 2.3. Write $K := \cap_j K_j$ and suppose that $\delta^H(K_i, K) \not\to 0$. Since K_i is a decreasing sequence, the there is a $\delta > 0$ so that $K_m \subseteq K + \delta B$ for all m (because $\inf_m \delta_H(K_m, K) > 0$, taking half of that infimum should suffice).

Take now $A_m = K_m \setminus \text{int}(K + \delta B)$. Note that (A_m) is decreasing and compact (because (K_m) is), so their intersection $A := \cap_m A_m$ is not empty, by the Intersection Theorem.

Note that since $A_m \subseteq K_m$, we have $A \subseteq K$. However $A \cap K = \emptyset$: if there is $x \in A \cap K$, then x belongs to K but not in the $\text{int}(K + \delta B)$. This brings the wanted contradiction.

$\square$

Now we're ready to prove Theorem 2.5.

Proof of Theorem 2.5: Let $(K_i) \subseteq \mathcal{C}^n$ be Cauchy in δ^H. Then (K_i) is a bounded in δ^H (i.e. the distance between any two bodies is bounded) and so does $\cup_i K_i$. That is, it is contained in a large enough multiple of the unit ball.

Now consider sequence of tails $A_m = \text{cl} \cup_{i=m}^{\infty} K_i$. Then A_m is compact (because the $\cup_i A_i$ is bounded and we take the closure of the tail) and (A_m) is a decreasing sequence. By the lemma above, (A_m) converges in δ^H to $\cap_m A_m =: A$.

We claim now that K_i converges to A. Indeed, fix $\varepsilon > 0$. By definition of convergence, we have $A_m \subseteq A + \varepsilon B$ for $m \geq n_0$ for some n_0. In particular, $K_i \subseteq A + \varepsilon B$ for $i \geq n_0$. Also note since (K_i) is Cauchy, we can choose $n_1 \geq n_0$ so that $\delta^H(K_i, K_j) < \varepsilon$ for $i, j \geq n_1$, i.e. $K_j \subseteq K_i + \varepsilon B$. Consequently, $\cup_{j=m}^{\infty} K_j \subseteq K_i + \varepsilon B$ for $i, m \geq n_1$.

Combining this with earlier observation, we know that $A_m \subseteq K_i + \varepsilon B$. In particular, the smaller set A should also sit in $K_i + \varepsilon B$ for all $i \geq n_1$. But this precisely is the definition of K_i converging to A, as claimed. To complete the proof then, we just need to note that A is also compact, hence the limit also belongs to $\mathcal{C}^n$.

$\square$

To complete this part, we'll prove the weak compactness property of these two spaces, as stated in the following theorem.

Theorem 2.7 (Blaschke Selection). *Each bounded sequence in $\mathcal{K}^n$ has a subsequence that converges (in δ^H) to an element of $\mathcal{K}^n$.*

As is probably expected, this property is inherited from the superseding space $\mathcal{C}^n$. Thus most of the work lies in proving the following theorem.

Theorem 2.8. *Every bounded sequence in $\mathcal{C}^n$ has a convergent subsequence.*

Proof. Let $(K_i^0)_i \subseteq \mathcal{C}^n$ be a bounded sequence. Then for all i, K_i^0 is contained in a centered cube C of side length γ. By the completeness of $\mathcal{C}^n$ (Theorem 2.5), it suffices then to extract a Cauchy sequence from $(K_i^0)_i$.

We will choose a subsequence as follows: for each m, divide C into a grid of side length $2^{-m} \cdot \gamma$. Define then $A_m(K)$ to be the union of sub-cubes of C of length $2^{-m} \cdot \gamma$ (from the grid above) that meet K. Now note that $(K_i^0)_i$ is infinite and the grid is composed only of finite sub-cubes, an application of the pigeon hole principle tells us that it is possible to find a subsequence $(K_i^1)_i$ (here we renumber the sequence if necessary) so that $A_1(K_i^1)$ is the same set for all i. Otherwise, each K_i^0 will have a unique grid covering, which is impossible for there are only 2^n possible sub-cubes to choose from.

Continue this process: for each m choose a subsequence $(K_i^{m+1})_i \subseteq (K_i^m)_i$ so that $(A_{m+1}(K_i^{m+1}))_i$ is a trivial sequence. Now we take the diagonal: let $K_m = (K_m^m)$. We claim that (K_m) is a Cauchy sequence.

First, observe that $A_m(K_m^m) \subseteq K_j^m + \lambda B$ for all j, where $\lambda = 2^{-m}\sqrt{n}\lambda$ (half of diagonal length of the m-th grid). This inclusion is true, because $A_m(K_j^m) = A_m(K_m^m)$ by construction and we just expand K_j^m by the maximum distance to the maximum reach of the grid's sub-cube (at the m-th level). In particular, because K_i^m is contained in $A_m(K_m^m)$ for each i, we have $K_i^m \subseteq K_j^m + \lambda B$ for all i, j, and m.

The last inclusion then implies that for all i, j, and m

$$\delta^H(K_i^m, K_j^m) \leq \frac{\sqrt{n}\gamma}{2^m}$$

In particular it holds as we go along the diagonal: since given any $m' \geq m$, $K_{m'} = K_j^m$ for some j, we know that

$$\delta^H(K_m, K_{m'}) \leq \frac{\sqrt{n}\gamma}{2^m}$$

for all $m' \geq m$. Hence the sequence (K_m) is Cauchy and the theorem follows from Theorem 2.5.

□

2.2 The Support Functions and Its Properties

Definition 2.9. For $K \in \mathcal{K}^n$, define the support function of K as

$$h_K(u) = \sup_{x \in K} \langle x, u \rangle \qquad u \in \mathbb{R}^n$$

The width is then defined as

$$w_K(u) = h_K(u) + h_K(-u) \qquad u \in \mathbb{R}^n$$

Lemma 2.10. *For* $K_1, K_2 \in \mathcal{K}^n$*,*

$$\delta^H(K_1, K_2) = \sup_{u \in S^{n-1}} |h_{K_1}(u) - h_{K_2}(u)|$$

This last characterization gives an alternative proof of the selection theorem (Theorem 2.7).
Another proof of Theorem 2.7:

Definition 2.11. Let $K \in \mathcal{K}^n$ and assume that $0 \in \text{int}K$. Then define the gauge functional with respect to K as

$$\|x\|_K = \inf\{\lambda \geq 0 \,|\, x \in \lambda K\}$$

Also define the radial function as

$$\rho_K(u) = \sup\{t \geq 0 \,|\, tu \in K\} \qquad u \in S^{n-1}$$

Definition 2.12. Let $K \in \mathcal{K}^n$ and $0 \in \text{int}K$. The polar body of K is

$$K^\circ = \{y \in \mathbb{R}^n \,|\, \langle x, y\rangle \leq 1 \text{ for each } x \in K\}$$

Equivalently, $K^\circ = \{y \in \mathbb{R}^n \,|\, h_K(y) \leq 1\}$.

Proposition 2.13. K° *is convex and* $(K^\circ)^\circ = K$.

Proof. □

Just like the correspondence between the norm on a space and its dual, we can find an analogous relation between the "norm" of a body and its polar.

Proposition 2.14. *Let* $K \in \mathcal{K}^n$ *with* $0 \in$ *int*K*. Then for* $x \in \mathbb{R}^n$*,* $\|x\|_K = h_{K^\circ}(x)$.

Proof. We'll show that both "norms" give the same "ball" (symmetric issue aside). In particular, homogeneity of the functions means it's enough to consider unit balls. So take first $\widetilde{K} = \{h_{K^\circ}(x) \leq 1\}$. Let now $x \in \widetilde{K}$. Then for any $y \in K^\circ$, we have $\langle x, y\rangle \leq h_{K^\circ}(x) \leq 1$, so $x \in (K^\circ)^\circ = K$ and $\widetilde{K} \subseteq \{\|x\|_K \leq\}$.
Conversely, given $x \in K$ (i.e. $\|x\|_K \leq 1$), by compactness we can find $v \in K^\circ$ that realizes the support function, i.e. so that $h_{K^0}(x) = \langle x, v\rangle$. From the definition of polar, the last quantity is bounded by 1, so $x \in \widetilde{K}$.

□

Note here that the proof doesn't require that K is symmetric. However if K is symmetric, $\|\cdot\|_K$ is a norm (it's now absolutely homogeneous) and hence we obtain a correspondence between origin-symmetric convex bodies in $\mathbb{R}^n$ and closed unit balls n-dimensional normed spaces.

2.3 Sections and Projections

Let $E \subseteq \mathbb{R}^n$ be a subspace. For any given body K, its section on E is the intersection $K \cap E$. In this part, we will explore the relation between the section with E and the (orthogonal) projection to E of a body K. The basic relation is given by the following proposition, which is a translation of the same fact in functional analysis.

Proposition 2.15. *Let $K \in \mathcal{K}^n$ with $0 \in intK$. Then for any subspace E,*

(a) $P_E K^\circ = (K \cap E)^\circ$. The polar of $K \cap E$ is in E.

(b) $(P_E K)^\circ = K^\circ \cap E$. The polar of $P_E K$ is taken on E.

Note. In functional analysis, replace the word section with subspace, projection with quotient, and polar with dual. See Proposition 3.51 in Nigel Kalton's Functional Analysis, for example.

Proof. To prove (a), we'll prove two inclusions. Consider first when $z \in K^\circ$. Let $y \in K \cap E$ and take its inner product with $P_E z$. Since y already lives in E, its inner product with $z - P_E z$ vanishes and hence $\langle P_E z, y\rangle = \langle z, y\rangle$. But y is also in K, so the last quantity is bounded by 1. Hence $z \in (K \cap E)^\circ$.

To prove the opposite, we'll recall that polar reverses inclusion, so we can show $(P_E K^\circ)^\circ \subseteq K \cap E$. Let $x \in (P_E K^\circ)^\circ$. Then for any given $y \in K^\circ$, $\langle x, P_E y\rangle \leq 1$. The polar is taken in E, so x also belongs in E. Then by the same reasoning as before, this means $\langle x, P_E y\rangle = \langle x, y\rangle \leq 1$. Since $y \in K^\circ$, this means $x \in (K^\circ)^\circ = K$ too, which gives the wanted inclusion.

The proof of (b) is similar.

□

2.4 More on Convex Sets

We'll delve more on convex sets and their properties. Before that, some definition is needed.

Definition 2.16. Let $x, x_1, \cdots, x_k \in \mathbb{R}^n$.

(a) x is an affine combination of $x_1, \cdots, x_k$ if

$$x = \sum_{i=1}^{k} \lambda_i x_i \text{ with } \sum_{i=1}^{k} \lambda_i = 1 \text{ and } \lambda_i \in \mathbb{R}$$

Call then aff(A) to be the collection of all affine combinations of finitely many elements of A.

(b) x is a convex combination of $x_1, \cdots, x_k$ if

$$x = \sum_{i=1}^{k} \lambda_i x_i \text{ with } \sum_{i=1}^{k} \lambda_i = 1 \text{ and } \lambda_i \geq 0$$

Call then conv(A) to be the collection of all convex combinations of finitely many elements of A. It is called the convex hull of A.

As a consistency check, we'll prove the following proposition.

Proposition 2.17. *Let $A, B \subseteq \mathbb{R}^n$.*

(a) If A is convex, then conv$A = A$.

(b) conv$A = \cap\{K \mid K$ is convex and contains $A\}$

Proof. We'll begin with (a). Clearly $A \subseteq$ conv$A$. We'll prove maximality by induction. Let $k > 2$ and assume that any convex combination of $(k-1)$ elements of A belongs to A. Now consider the convex combination $x = \sum_{i=1}^{k} \lambda_i x_i \in$ conv$A$. We may assume here $\lambda_i > 0$, otherwise we're back to the previous case.
Rewrite the linear combination now as

$$x = (1 - \lambda_k) \sum_{i=1}^{k-1} \frac{\lambda_i}{1 - \lambda_k} x_i + \lambda_k x_k \tag{1}$$

Observe that $\frac{\lambda_i}{1-\lambda_k} > 0$ and their sum from 1 to $k-1$ is 1. By the induction hypothesis then, $\sum_{i=1}^{k-1} \frac{\lambda_i}{1-\lambda_k} x_i$ belongs to A. But then (1) is just a convex sum of 2 elements in A. Hence $x \in A$.
For brevity, denote the intersection in (b) as K_A. It's easy to see that $A \subseteq K_A$, so conv$A \subseteq$ conv$K_A = K_A$ by (a). On the other hand, $K_A \subseteq$ conv$A$ (because conv$A$ is one of the set in the intersection). This proves the opposite inclusion. □

Note only the action of taking the convex hull is stable, it can also inherit the compactness of a finite union of compact sets.

Proposition 2.18. *Let $K_1, \cdots, K_m \in \mathcal{K}^n$. Then*

$$\textit{conv}(K_1 \cup \cdots \cup K_m) = \left\{ \sum_{i=1}^{m} \lambda_i x_i \mid x_i \in K_1,\ \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

In particular, conv$(K_1 \cup \cdots \cup K_m) \in \mathcal{K}^n$.

Proof. Observe that by definition

$$A_1 := \operatorname{conv}(K_1, \cdots, K_m) = \left\{ \sum_{i=1}^{m} \lambda_i x_i \,|\, x_i \in K_1, \lambda_i \geq 0, i = 1, \cdots, m, \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

Since $K_1, \cdots, K_m$ belong in their union, it is clear that

$$A_1 \subseteq \operatorname{conv}\left(\bigcup_{i=1}^{m} K_i \right) =: A_2$$

To show the opposite direction, consider an arbitrary element of A_2, which can be written as

$$\sum_{j=1}^{m'} \alpha_j y_j \text{ where } \alpha_j \geq 0 \; \forall j = 1, \cdots, m', \; \sum_{j=1}^{m'} \alpha_j = 1, \text{ and } y_j \in \bigcup_{i=1}^{m} K_i \tag{2}$$

After possible renumbering, consider the partitions $J_i = \{j_i, \cdots, j_{i+1} - 1\}$ of $\{1, \cdots, m'\}$, $i = 1, \cdots, m$, so that for each $j \in J_i$, y_j belongs to K_i and $y_{j_1} = 1$. For example, if $y_1, \cdots, y_{j_2-1}$ belong to K_1, $y_{j_2}, \cdots, y_{j_3-1}$ belong to K_2, and so on. If a point belongs to multiple K_i's, we'll choose the one with smallest index. Of course, some of the J_i's are allowed to be empty.

After grouping, let's analyse the partition J_1: suppose $y_1, \cdots, y_j \in K_1$. Then note that by convexity of K_1,

$$\frac{\alpha_1}{\alpha_1 + \cdots + \alpha_j} y_1 + \cdots + \frac{\alpha_j}{\alpha_1 + \cdots + \alpha_j} y_j \in K_1$$

Call that convex combination x_1. Also because each $\alpha_j \geq 0$, we know $0 \leq \lambda_1 := \sum_{j \in J_1} \alpha_j \leq 1$. Repeat the same observation for each (non-empty) grouping J_i and define x_i and λ analogously. Then we can rewrite (2) as

$$\sum_{j=1}^{m'} \alpha_j y_j = \sum_{i=1}^{m} \lambda_i x_i$$

where $\lambda_i \geq 0$, $\sum_{i=1}^{m} \lambda_i = \sum_{j=1}^{m'} \alpha_j = 1$, and $x_i \in K_i$ for each $i = 1, \cdots, m$. Since we can represent any element of A_2 this way, we have the opposite inclusion and may conclude that $A_1 = A_2$, as wanted.

To then show that A_1 is compact, consider the map

$$K_1 \times \cdots \times K_m \times D_m \to \operatorname{conv}\left(\bigcup_{i=1}^{m} A_i \right)$$

$$(x_1, \cdots, x_m, (\lambda_i)_{i=1}^{m}) \mapsto \sum_{i=1}^{m} \lambda_i x_i$$

where D_n is the simplex on the positive octant. It's not hard to see that this map is surjective (because $A_1 = A_2$) and continuous (because it is linear on the first n position and continuously change on the last position) with the product topology. Then we find that A_2 is a continuous image of a compact set (finite product of compact sets is compact), so A_2 is also compact, as claimed. □

2.5 Approximation by Polytope

The objects in $\mathcal{K}^n$ might be abstract, but they can be approximated with much simpler sets. For $\mathcal{K}^n$, such approximating object is the polytopes.

Definition 2.19. A set $P \subseteq \mathbb{R}^n$ is a polytope if $P = \text{conv}T$ where T is a finite set.

Proposition 2.20. *Let $K \in \mathcal{K}^n$ and $\varepsilon > 0$. Then there is a polytope $P \in \mathcal{K}^n$ such that $P \subseteq K \subseteq P + \varepsilon B$. In particular, $\delta^H(P, K) < \varepsilon$.*

Proof. Fix $\varepsilon > 0$. By compactness, we can cover K by ε-balls centered at some $x_1, \cdots, x_N \in K$. Take then $P = \text{conv}\{x_1, \cdots, x_n\}$. Easily $P \subseteq K \subseteq \cup_{i=1}^N x_i + \varepsilon B$. Moreover, the rightmost set is contained in $\text{conv}\{x_1, \cdots, x_N\} + \varepsilon B = P + \varepsilon B$, which completes the proof. □

References

[1] Schneider, Rolf; *Convex Bodies: The Brunn-Minkowski Theory*, 2013