Cauchy Inequalities

Paul Simanjuntak

28/02/2020

Note. Most of this talk is taken (in pieces) from "The Cauchy-Schwarz Master Class" by J. Michael Steele. The figure is also taken from the book.

Probably the most widely-used (and well-known, and beloved, and other superlative adjective that one can attach to illustrate how pervasive this little identity is) expression in analysis, the Cauchy inequality relates the sum of products to product of sums. In the most basic form, it can be stated as follows:

Theorem 1 (Cauchy inequality, 1821)**.** *Given real sequences $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$, we have*

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \tag{1}$$

This little statement was first published by Cauchy himself and made famous, at least in the Russian-speaking circles, by his student Bunyakovsky (interestingly, Bunyakovsky never bothered to deal with the infinite series, claiming it is an "obvious" extension of the finite version). For some reason, this work was not well-known in Western Europe until much later, because manyy years later in Gottingen, Schwarz independently proved the integral analog of the finite sum version:

Theorem 2 (Cauchy-Schwarz inequality, 1885)**.** *If $S \subseteq \mathbb{R}^2$ and $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$, then the double integrals*

$$A = \iint_S f^2 \, dx \, dy \qquad B = \iint_S f g \, dx \, dy \qquad C = \iint_S g^2 \, dx \, dy$$

must satisfy

$$|B| \leq \sqrt{A} \cdot \sqrt{C}$$

This talk does not have an overarching goal, except that it will all be related to the Cauchy inequality. We will look at an interesting way this expression come to be and obtain inequality through a useful (but seldom discussed) tool. We will end with the question that will stump many student: when does the backward inequality hold?

As is customary, we should begin by proving Theorem 1 (Note: for the rest of this talk, we'll just concern ourselves with finite real sequences). The proof will be by induction. The case when $n = 1$ is not interesting, so we'll start with $n = 2$. If one tries to write out (1), one gets

$$(a_1 b_1 + a_2 b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2) \tag{2}$$

It is not hard (but a bit annoying) to check the inequality above by expanding both sides. Another way is to notice that the inequality is a consequence of

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) = (a_1 b_1 + a_2 b_2)^2 + (a_1 b_2 - a_2 b_1)^2 \tag{3}$$

which is always non-negative. Dropping the first term on the right side of () will give Equation (1) for $n = 2$. Avoiding expansion of (2) by introducing () seems pointless. There is (as always) a better way: with the figure below, use Pythagorean identity

$$\cos^2(\alpha + \beta) + \sin^2(\alpha + \beta) = 1$$

and use sum-of-angle identity to express everything in elementary angles. Substitute the trigonometeric functions with the corresponding sides' ratios to get (). This proves the base case. From here, the induction step is quite simple, so we will leave it to the reader.

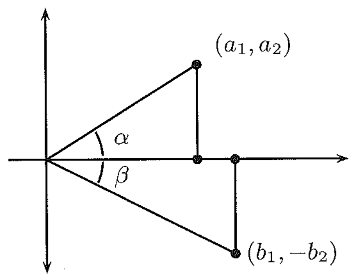

Figure 1: THE figure below

Let's scrutinize again (). We completely ignored the term $(a_1 b_2 - a_2 b_1)^2$ and just used the fact that it is non-negative. While this gives what we needed, one might ask, how much information did we lose from the castaway term(s)?
It turns out to be an important question, because we want to know how much defect is incurred after each application of (1). To quantify this, we'll look at the difference of the two sides in the Cauchy inequality: define the defect Q_n as

$$Q_n = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) - \left(\sum_{i=1}^{n} a_i b_1\right)^2 \tag{4}$$

which after cancellation will yields

$$Q_n = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2 b_j^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_i a_j b_j$$

Note that in the second double sum, the roles of i and j are symmetric. To make the first double sum looks similar, we can write

$$\begin{aligned} Q_n &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i^2 b_j^2 + a_j^2 b_i^2) - \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_i a_j b_j \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 \end{aligned} \tag{5}$$

which is the ”castaway term” we saw at the beginning of this discussion. Comparing (4) and (5), we have what is called Lagrange’s identity:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2$$

A quick application of Lagrange’s identity is to give equality in Theorem 1: we must then have $Q_n = 0$, so for fixed k and $1 \le i \le n$, we have

$$a_i b_k = a_k b_i \qquad \Rightarrow \qquad \frac{a_i}{b_i} = \text{constant for } 1 \le i \le n$$

So we have just shown that equality in (1) is achieved when a is a multiple of b using this concept of defect. As a matter of fact, we can use similar idea to derive other (finite-dimensional real) inequalities and their equality cases.

Last, we’ll end with this question: when can (1) go in the other direction? At first glance this seems hard (just look at). On the other hand, it is actually not very sharp (compare two orthogonal vectors, say). To make the question explicit, we’re looking for a constant $\rho > 0$ so that

$$\left(\sum_{k=1}^{n} a_k^2\right)^{1/2}\left(\sum_{k=1}^{n} b_k^2\right)^{1/2} \le \rho\sum_{k=1}^{n} a_k b_k$$

Let’s begin with the simplest non-trivial case: with vectors $(1, a)$ and $(1, b)$, when $a, b > 0$. Then we’ll compare

$$(1 + a^2)^{1/2}(1 + b^2)^{1/2} \le \rho(1 + ab)$$

Observe: if we keep the product ab constant but letting $a \to \infty$, the left side will diverge while the right side stays the same. This suggests that we should bound the ratio, so let’s impose the condition

$$m \le \frac{a_k}{b_k} \le M \qquad k = 1, 2, \cdots, n$$

for some $0 \le m \le M < \infty$.

This is all nice and good, but how can we get ρ out of this? If we’re to follow our nose, we’ll try the easy bound

$$\left(M - \frac{a_k}{b_k}\right)\left(\frac{a_k}{b_k} - m\right) \ge 0$$

and expand it to get for all k that

$$a_k^2 + (mM)b_k^2 \le (m + M)a_k b_k$$

Adding all these, we obtain that

$$\sum_{k=1}^{n} a_k^2 + (mM)\sum_{k=1}^{n} b_k^2 \le (m + M)\sum_{k=1}^{n} a_k b_k$$

This is almost it. To relate the product with sum, we'll need an elementary identity used in the standard proof of (1): for $a, b > 0$,

$$XY \leq \frac{1}{2}(X^2 + Y^2)$$

Hence we conclude

$$\begin{aligned}\left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(mM \sum_{k=1}^{n} b_k^2\right)^{1/2} &\leq \frac{1}{2}\left(\sum_{k=1}^{n} a_k^2 + (mM)\sum_{k=1}^{n} b_k^2\right) \\ &\leq \frac{1}{2}(m+M)\sum_{k=1}^{n} a_k b_k\end{aligned}$$

The result can put in the following statement.

Theorem 3 (Reverse Cauchy inequality)**.** *Let* $a_k, b_k \geq 0$ *for all* $k = 1, \cdots, n$ *with*

$$0 < m \leq \frac{a_k}{b_k} \leq M < \infty$$

Define

$$A = \frac{1}{2}(m+M) \qquad G = \sqrt{mM}$$

Then we have

$$\left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} b_k^2\right)^{1/2} \leq \frac{A}{G}\sum_{k=1}^{n} a_k b_k$$