# LECTURE NOTES: STEIN-TOMAS RESTRICTION THEOREM AND STRICHARTZ ESTIMATES FOR SCHRÖDINGER'S EQUATION

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## 1. Stein-Tomas Restriction Theorem

We recall the following decay estimate for the Fourier transform.

**Theorem 1.1.** *Suppose that $S \subset \mathbb{R}^{n+1}$ is a smooth hyper-surface with everywhere non-vanishing Gaussian curvature, and suppose that $d\mu = \psi d\sigma$, where $\psi \in C_0^\infty$, and $d\sigma$ denotes surface measure on $S$. Then*

$$\widehat{\mu}(X) \leq C|X|^{-n/2},$$

*where $\hat{}$ denotes the Fourier transform.*

In this note we will prove the following pair of Theorems.

**Theorem 1.2. (Stein - Tomas restriction theorem).** *Suppose that $S \subset \mathbb{R}^{n+1}$ is a smooth hyper-surface with everywhere non-vanishing Gaussian curvature, and suppose that $d\mu = \psi d\sigma$, where $\psi \in C_0^\infty$, and $d\sigma$ denotes surface measure on $S$. Then for $1 \leq p \leq p_{n+1} := (2n+4)/(n+4)$, we have*

$$\left(\int_S |\widehat{f}(\zeta)|^2 d\mu(\zeta)\right)^{1/2} \lesssim \|f\|_{L^p(\mathbb{R}^{n+1})}, \tag{1.3}$$

*where the implicit constant depends upon $p, n, \psi$ and $S$.*

**Theorem 1.4. (Strichartz estimate for Schrödinger's equation).** *Suppose that $u = u(x,t)$ solves the initial value problem for Schrodinger's equation:*

$$\begin{cases} iu_t + \Delta u = 0 \text{ in } \mathbb{R}^{n+1}_+ \\ u(x,0) = g(x) \in L^2(\mathbb{R}^n). \end{cases} \tag{IVPS}$$

*Then*

$$\|u\|_{L^{q_s}(\mathbb{R}^{n+1}_+)} \leq C\|g\|_{L^2(\mathbb{R}^n)}, \tag{1.5}$$

*where $q_s \equiv (2n+4)/n$.*

We remark that more general results (namely, for "mixed norm" spaces) are true, and in fact, our proof below will yield them.

We consider first Theorem 1.2.

*Proof.* By a partition of unity, we may suppose that $\operatorname{supp}\psi \cap S$ is contained in a co-ordinate patch of $S$, on which $S$ coincides with a graph $\tau = \varphi(\xi)$, where

$\varphi \in C^\infty(\mathbb{R}^n)$ and has non-vanishing Hessian matrix. Then $d\sigma = \sqrt{1+|\nabla\varphi(\xi)|^2}d\xi$ (at least locally), and

$$\int_S |\widehat{f}(\zeta)|^2 d\mu(\zeta) = \int_{\mathbb{R}^n} |\hat{f}(\xi,\varphi(\xi))|^2\,\psi(\xi,\varphi(\xi))\,\sqrt{1+|\nabla\varphi(\xi)|^2}\,d\xi \tag{1.6}$$
$$= \int \hat{f}(\xi,\varphi(\xi))\overline{\hat{f}(\xi,\varphi(\xi))}\,\widetilde{\psi}(\xi)\,d\xi,$$

where $\widetilde{\psi}(\xi) := \psi(\xi,\varphi(\xi))\sqrt{1+|\nabla\varphi(\xi)|^2}$. Writing out the Fourier transform, this equals (up to a multiplicative constant)

$$\int \left(\iint e^{-i(x\cdot\xi+t\varphi(\xi))} f(x,t)dxdt\right)\left(\iint e^{i(y\cdot\xi+s\varphi(\xi))}\,\overline{f(y,s)}dyds\right)\widetilde{\psi}(\xi)\,d\xi \tag{1.7}$$
$$= \iint f(x,t)\iint\left[\int e^{-i(x-y)\cdot\xi-i(t-s)\varphi(\xi)}\,\widetilde{\psi}(\xi)\,d\xi\right]\overline{f(y,s)}\,dyds\,dxdt$$
$$= \iint f(x,t)\left(\iint K_{t-s}(x-y)\overline{f(y,s)}dyds\right)dxdt$$
$$=: \iint f(x,t)\int T_{t-s}\overline{f(\cdot,s)}(x)\,ds\,dxdt$$

where

$$T_{t-s}g(x) := K_{t-s} * g(x),$$

and

$$K_{t-s}(x) := C\int_{\mathbb{R}^n} e^{-ix\cdot\xi-i(t-s)\varphi(\xi)}\,\widetilde{\psi}(\xi)\,d\xi = \mathcal{F}\left(e^{-i(t-s)\varphi(\xi)}\,\widetilde{\psi}(\xi)\right)(x), \tag{1.8}$$

and where for purposes of legibility we haved used an alternative notation for the $n$-dimensional Fourier Transform acting in the space variables only:

$$\mathcal{F}(h) := \widehat{h} \quad \text{in } \mathbb{R}^n.$$

We now record several observations. First, recalling the definition of $\widetilde{\psi}$, we see that $K_{t-s}(x-y) = \widehat{\mu}(x-y,t-s)$, so that by Theorem 1.1 we have

$$|K_{t-s}(x-y)| \le C|(x-y,t-s)|^{-n/2} \le C|t-s|^{-n/2}. \tag{1.9}$$

Consequently,

$$\|T_{t-s}g\|_{L^\infty(\mathbb{R}^n)} \le C|t-s|^{-n/2}\|g\|_{L^1(\mathbb{R}^n)}\,. \tag{1.10}$$

Moreover,

$$\mathcal{F}(K_{t-s})(\xi) = \mathcal{F}^{-1}(K_{t-s})(-\xi) = e^{-i(t-s)\varphi(-\xi)}\,\widetilde{\psi}(-\xi)\,,$$

where in the last step we have used (1.8). Thus $\|\mathcal{F}(K_{t-s})\|_\infty \le \|\widetilde{\psi}\|_\infty$, so that by Plancherel,

$$\|T_{t-s}g\|_{L^2(\mathbb{R}^n)} = C\,\|\mathcal{F}(K_{t-s})\,\widehat{g}\|_{L^2(\mathbb{R}^n)} \le C\,\|\widetilde{\psi}\|_\infty\|g\|_{L^2(\mathbb{R}^n)}\,. \tag{1.11}$$

Interpolating between (1.10) and (1.11) we obtain

$$\|T_{t-s}g\|_{L^{r'}(\mathbb{R}^n)} \le C\,|t-s|^{-\beta}\,\|g\|_{L^r(\mathbb{R}^n)}\,, \qquad r = \frac{2}{1+\theta},\ \beta = \frac{\theta n}{2}\,, \tag{1.12}$$

where $0 \leq \theta \leq 1$, since $r^{-1} = (1-\theta)/2 + \theta$ is equivalent to $r = 2/(1+\theta)$. Moreover, by the H-L-S Theorem, the 1-dimensional fractional integral operator

$$I_{1-\beta}\, h(t) := c \int_{\mathbb{R}} |t-s|^{-\beta} h(s) ds$$

satisfies

$$\|I_{1-\beta}\, h\|_{L^{p'}(\mathbb{R})} \leq C\|h\|_{L^p(\mathbb{R})}\,, \qquad p = \frac{4}{4-n\theta}\,, \tag{1.13}$$

where $(p-1)/p = (p')^{-1} = p^{-1} - 1 + \beta$ and $\beta = \theta n/2$. Indeed, solving for $p$, we obtain $p = 2/(2-\beta) = 4/(4-n\theta)$.

We define the mixed norm space $L^p_t L^r_x$ as the space of measurable functions on $\mathbb{R}^{n+1}$ for which the norm

$$\|u\|_{L^p_t L^r_x} \equiv \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} |u(x,t)|^r dx \right)^{p/r} dt \right)^{1/p}$$

is finite. We now claim that the following "Generalized Stein-Tomas estimate" holds:

$$\left( \int_S |\widehat{f}|^2 d\mu \right)^{1/2} \lesssim \|f\|_{L^p_t L^r_t}\,, \qquad r = \frac{2}{1+\theta}\,,\ \ p = \frac{4}{4-n\theta}\,,\ \ 0 \leq \theta < \frac{2}{n}\,, \tag{1.14}$$

where the implicit constant depends upon $n, \psi, \mathcal{S}, p$ and $r$. We note that Theorem will follow, once we have established the claim. Indeed, $r = p$ when $\theta = 4/(4+2n)$ and $p = p_{n+1} = (2n+4)/(n+4)$, which is Theorem 1.2.

We establish the claim as follows. Combining (1.6) and (1.7) we obtain

$$\begin{aligned}
\int_S |\widehat{f}|^2 d\mu &= \iint_{\mathbb{R}^{n+1}} f(x,t) \int_{\mathbb{R}} T_{t-s}\overline{f(\cdot,s)}(x)\, ds\, dxdt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} f(x,t) T_{t-s}\overline{f(\cdot,s)}(x)\, dx \right) dsdt \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \|f(\cdot,t)\|_{L^r(\mathbb{R}^n)}\, \|T_{t-s}\overline{f(\cdot,s)}\|_{L^{r'}(\mathbb{R}^n)}\, dsdt \\
&\lesssim \int_{\mathbb{R}} \|f(\cdot,t)\|_{L^r(\mathbb{R}^n)} \int_{\mathbb{R}} |t-s|^{-\beta}\, \|\overline{f(\cdot,s)}\|_{L^r(\mathbb{R}^n)}\, dsdt\,,
\end{aligned}$$

where in the second line we have used Fubini's Theorem, in the third line Hölder's inequality, and in the last line (1.12). Setting $h(t) = \|f(\cdot,t)\|_{L^r(\mathbb{R}^n)}$, we have that the last line equals (up to a multiplicative constant)

$$\int_{\mathbb{R}} h(t) \left( I_{1-\beta} h \right)(t)\, dt \leq \|h\|_{L^p(\mathbb{R})}\, \|I_{1-\beta} h\|_{L^{p'}(\mathbb{R})} \lesssim \|h\|^2_{L^p(\mathbb{R})}$$

by (1.13). By definition, $\|h\|_{L^p(\mathbb{R})} = \|f\|_{L^p_t L^r_x}$, and the claim follows. □

## 2. Schrödinger's equation

*Proof.* Observe that, up to a normalization, the Fourier transform (in the space variable) of the solution to IVPS is given by

$$\hat{u}(\xi,t) = e^{-it|\xi|^2}\hat{g}(\xi). \tag{2.1}$$

Indeed, then

$$i\,\frac{\partial \hat{u}}{\partial t} = i(-i|\xi|^2)\hat{u} = |\xi|^2\hat{u} = C(\widehat{\Delta u}).$$

Taking inverse transforms, we see that u solves IVPS, except for the normalization; the convergence to the initial data holds in the $L^2$ norm, by Plancherel. Note that the solution can be extended to the lower half space in the same way, so that we might as well consider our solution to exist in all of $\mathbb{R}^{n+1}$. To prove estimate (1.5), we dualize. Let $q \equiv q_s$, and let $p = q/(q-1)$ be the dual exponent. Note that we then have $p = p_{n+1} = (2n+4)/(n+4)$. More generally, we shall prove

$$\|u\|_{L_t^{p'} L_x^{r'}} \leq C(p,r)\, \|g\|_{L^2(\mathbb{R}^n)}, \tag{2.2}$$

where $p, r$ are the same exponents as in (1.14). Let $h \in L_t^p L_x^r$, and consider

$$\iint h(x,t)u(x,t)dxdt = C \iint h(x,t) \int e^{ix\cdot\xi} e^{-it|\xi|^2} \hat{g}(\xi) d\xi dxdt$$

(where we have expressed $u$ as the inverse Fourier transform of (1.6)). By Fubini, this equals

$$\int \hat{g}(\xi) \left( \iint e^{-i\left(x\cdot\xi + t|\xi|^2\right)} h(-x,t)dxdt \right) d\xi$$

(where we have changed variables $x \to -x$), which in turn equals

$$\int \hat{g}(\xi) \widehat{f}(\xi, |\xi|^2) d\xi,$$

where $f(x,t) \equiv h(-x,t)$. By Cauchy-Schwartz, the last expression is bounded in absolute value by

$$\|g\|_2 \left( \int_{\mathbb{R}^n} |\widehat{f}(\xi, |\xi|^2)|^2 d\xi \right)^{1/2}.$$

Thus, it is enough to prove that

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi, |\xi|^2)|^2 d\xi \leq \|f\|^2_{L_t^p L_x^r}. \tag{2.3}$$

Estimate (2.3) is the "Stein-Tomas" restriction estimate for the paraboloid. We first note that

$$\int_{\frac{1}{2} \leq |\xi| \leq 2} |\widehat{f}(\xi, |\xi|^2)|^2 d\xi \leq C \|f\|^2_{L_t^p L_x^r} \tag{2.4}$$

Indeed, setting $\varphi(\xi) := |\xi|^2$, $d\sigma(\xi) := \sqrt{1 + |\nabla\varphi(\xi)|^2}\, d\xi$, $\psi(\xi) := \Phi(|\xi|)/\sqrt{1 + |\nabla\varphi(\xi)|^2}$, with $\Phi \in C_0^\infty(1/4, 4)$, $0 \leq \Phi \leq 1$ and $\Phi \equiv 1$ on $(1/2, 2)$, and $\psi\, d\sigma =: d\mu$, we see that

$$\int_{\frac{1}{2} \leq |\xi| \leq 2} |\widehat{f}(\xi, |\xi|^2)|^2 d\xi \lesssim \int_S |\widehat{f}|^2\, \psi\, d\mu,$$

where $S$ is the parabaloid $\tau = |\xi|^2$, so that ((2.4) follows immediately from (1.14). We now claim that

$$\int_{2^{k-1} \leq |\xi| \leq 2^{k+1}} |\widehat{f}(\xi, |\xi|^2)|^2 d\xi \leq C \|f\|^2_{L_t^p L_x^r}, \tag{2.5}$$

uniformly in $k \in \mathbb{Z}$. Let us assume momentarily that (2.5) holds. We choose a smooth partition of unity $1 = \sum_{k=-\infty}^{\infty} (\eta_k(\xi))^2$, where $\eta_k(\xi) = \eta(|\xi|/2^k)$ and $\eta \in$

$C_0^\infty(1/2, 2)$, with $0 \le \eta \le 1$. Set $\widehat{f_k} := \widehat{f}\eta_k$, so that $|\widehat{f}|^2 = \sum |\widehat{f_k}|^2$. Then

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi, |\xi|^2)|^2 d\xi = \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} |\widehat{f_k}(\xi, |\xi|^2)|^2 d\xi$$
$$\le C \sum_{k=-\infty}^{\infty} \|f_k\|^2_{L_t^p L_x^r} \le C \| \left( \sum_{k=-\infty}^{\infty} |f_k|^2 \right)^{1/2} \|^2_{L_t^p L_x^r}$$
$$\le C\|f\|^2_{L_t^p L_x^r},$$

where in the second line we have used first the claim (2.5) and then Minkowski's inequality (since $p, r \le 2$), and in the third line we have used the standard discrete Littlewood-Paley estimate in the $x$ variable (since $f_k = f * \check{\eta}_k =: Q_k f$, acting in the x variable).

Finally, to prove the claim, we observe that by the change of variable $\xi \to 2^k \xi$, the left hand side of (2.5) equals

$$\int_{\frac{1}{2} \le |\xi| \le 2} |\widehat{f^k}(\xi, |\xi|^2)|^2 d\xi \le \|f^k\|^2_{L_t^p L_x^r},$$

by (2.4), where $\widehat{f^k}(\xi, |\xi|^2) = 2^{kn/2} \widehat{f}(2^k \xi, |2^k \xi|^2)$. It is therefore enough to show that $\|f^k\|^2_{L_t^p L_x^r} = \|f\|^2_{L_t^p L_x^r}$. To this end, we observe that by definition

$$\widehat{f^k}(\xi, |\xi|^2) = 2^{kn/2} \widehat{f}(2^k \xi, |2^k \xi|^2)$$
$$= 2^{kn/2} \iint_{\mathbb{R}^{n+1}} e^{-i2^k \xi \cdot x} e^{-i|2^k \xi|^2 t} f(x, t)\, dxdt$$
$$= 2^{-k(2+n/2)} \iint_{\mathbb{R}^{n+1}} e^{-i\xi \cdot x} e^{-i|\xi|^2 t} f\left(2^{-k} x, 2^{-2k} t\right)\, dxdt,$$

where in the last line we have used the change of variable $x \to x/2^k$, $t \to t/2^{2k}$; i.e.,

$$f^k(x, t) = 2^{-k(2+n/2)} f\left(2^{-k} x, 2^{-2k} t\right).$$

Consequently,

$$\|f^k\|^2_{L_t^p L_x^r} = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} |f\left(2^{-k} x, 2^{-2k} t\right)|^r dx \right)^{p/r} dt \right)^{1/p} 2^{-k(2+n/2)}$$
$$= 2^{kn/r} 2^{2k/p} 2^{-k(2+n/2)} \|f\|_{L_t^p L_x^r},$$

where in the last line we have made the change of variable $x \to 2^k x$, $t \to 2^{2k} t$. Plugging in $r = 2/(1 + \theta)$, $p = 4/(4 - n\theta)$, we obtain after a computation that the exponent $k[n/r + 2/p - (2 + n/2)] = 0$. □