NOTES - 8631

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Remark. This is my (slightly more complete, but with far less foresight) notes for Harmonic Analysis II, as taught by Steve Hoffman in University of Missouri. Some of the terms and results referred can be found in "Harmonic Analysis I".
Whenever the sections and theorems cited have no link, the reader should refer back the previous course's notes.

1 Preliminaries

In this course, we will look at singular integral operators (SIOs) with non-convolution type kernels. In the previous course, we have discussed the convolution, which can be treated using methods of Fourier transform (see Section 5 of the previous notes). The prototypical example in those cases are the Hilbert and Riesz transforms. For the non-convolution type, the main example(s) come from the Calderon commutators.

Definition 1.1. Let $f : \mathbb{R} \to \mathbb{R}$ and let $A : \mathbb{R} \to \mathbb{R}$ be Lipschitz. Define the first-order Calderon commutator on $\mathbb{R}$ by

$$C^{(1)} f(x) := \frac{1}{2\pi i} \text{ p.v.} \int_{\mathbb{R}} \frac{1}{x-y} \left(\frac{A(x) - A(y)}{x-y} \right) f(y)\, dy$$

The k-th order Calderon commutator on $\mathbb{R}$ can be defined as

$$C^{(k)} f(x) = \frac{1}{2\pi i} \text{ p.v.} \int_{\mathbb{R}} \frac{1}{x-y} \left(\frac{A(x) - A(y)}{x-y} \right)^k f(y)\, dy$$

Notice that the kernel of this transform satisfies the C-Z conditions:

(i) $|K(x,y)| \leq \dfrac{C}{|x-y|^n}$

(ii) $\begin{matrix} |K(x+h,y) - K(x,y)| \\ |K(x,y+h) - K(x,y)| \end{matrix} \leq C \dfrac{|h|^\alpha}{|x-y|^{n+\alpha}}$ for some $\alpha \in (0,1]$ given $|x-y| \geq 2|h|$.

Note that

Lemma 1.2 (Fefferman-Stein). *Let b be in BMO class and $\{\psi_t\}_{t>0}$ be a family of kernels satisfying Littlewood-Paley assumptions so that for all $t > 0$,*

$$\Theta_t(x) = \int_{\mathbb{R}^n} \psi_t(x,y) f(y)\, dy = 0$$

Then $d\mu(x,t) = |\Theta_t b(x)|^2\, dx\, \frac{dt}{t}$ is a Carleson measure.

2 T1 theorem

Theorem 2.1 (Peetre/Spanne/Stein)**.** *Suppose T is a SIO with C-Z kernel that is L^2-bounded. Then T maps L^∞ to BMO with the estimate*

$$\|Tf\|_{BMO} \lesssim \|f\|_\infty$$

Theorem 2.2 (David-Journé T1 theorem)**.** *Let T be a SIO with C-Z kernel. Then the following are equivalent:*

(a) Both $T1$ and T^*1 are in BMO. Moreover, T satisfies the Weak Boundedness Property (WBP).

(b) T is L^2 bounded.

Theorem 2.3 (Christ-Journé $T1$ theorem)**.** *Let $\Theta_t f(x) = \int_{\mathbb{R}^n} \psi_t(x,y)f(y)\,dy$ where $\{\psi_t\}_{t>0}$ is of L-P classs. Assume $d\mu(x,t) = |\Theta_t 1(x)|^2\,dx\,\frac{dt}{t}$ is a Carleson measure. Then*

$$\int_{\mathbb{R}^n} (g_\psi f)^2\,dx \le C\,\|f\|_2^2$$

Lemma 2.4. *Let $Q_t f = \zeta_t * f$ for ζ some radial approximate identity with average 0 and supported on $B_{1/2}(0)$. Then if we define*

$$P_t := \int_t^\infty Q_s^2 \cdot \frac{ds}{s}$$

we have $P_t f$ is also an "approximate identity", i.e. $P_t f = \varphi_t * f$ with φ_t of similar property with ζ_t but supported on $B_1(0)$.

Theorem 2.5 (Local T1 theorem for SIO, first pass)**.** *Let T be a SIO with C-Z kernel. Suppose there exists an uniform constant C_0 so that for every cube Q, there exists $\eta_Q \in C_c^\infty(5Q)$, where $0 \le \eta_Q \le 1$ and $\eta_Q \equiv 1$ on $3Q$, with the estimate*

$$⨍_Q |T\eta_Q| + ⨍_Q |T^*\eta_Q| \le C_0$$

Then T is L^2-bounded.

Theorem 2.6 (Local T1 theorem, second pass)**.** *Let T be a SIO with C-Z kernel. Suppose there exists an uniform constant C_0 so that for all cube Q,*

$$⨍_Q (|T\chi_Q| + |T^*\chi_Q|) \le C_0$$

Then T is L^2-bounded.

3 Calderon transform and Lipschitz graph

Theorem 3.1 (Calderon)**.** *There exists a constant $C_0 > 0$ and an universal constant $C > 0$ so that*

$$\left\|C^{(k)} f\right\|_2 \leq C \cdot C_0^k \left\|A'\right\|_\infty^k \|f\|_2$$

Lemma 3.2 (Cotlar's inequality)**.** *If T is a SIO with C-Z kernel that is bounded, then*

$$T_* f := \sup_{\varepsilon>0} |T_\varepsilon f| = \sup_{\varepsilon} \left| \int_{|x-y|\geq\varepsilon} K(x,y) f(y)\, dy \right|$$

is L^2-bounded.

Corollary 3.3 (Calderon)**.** *Let $\Gamma = \{(x, A(x))\}$ is a Lipschitz graph. Define*

$$C_\Gamma f(z) = -\frac{1}{2\pi i}\, p.v. \int_\Gamma \frac{1}{z - v} f(v)\, dv \tag{1}$$

Then C_Γ is L^2-bounded of $\|A'\|_\infty \leq \varepsilon$ for some $\varepsilon > 0$.

Theorem 3.4 (Coifman/McIntosh/Meyer)**.** *Let C_Γ be defined as in* (1)*. Then C_Γ is L^2-bounded with no restriction on $\|A'\|_\infty$.*

3.1 Proof of Theorem 3.4 via square functions

Lemma 3.5 (Cotlar's inequality, equivalent form)**.** *Let T be a SIO with C-Z kernel and suppose T is L^2-bounded. Suppose*

$$T_* f(x) := \sup_{\varepsilon>0} |T_\varepsilon f(x)| = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} K(x,y) f(y)\, dy \right|$$

Then for all $s > 0$ and all $x \in \mathbb{R}^n$,

$$|T_* f(x)| \leq C_s \cdot M f(x) + \left(M|Tf|^s\right)^{1/s}(x)$$

In particular, T_* is of type $w(1,1)$ and L^p-bounded for $p > 1$.

Lemma 3.6 (Kolmogorov inequality)**.** *Let $0 < s < 1$ and g be weak L^1, i.e.*

$$\sup_{\lambda>0} \lambda\, |\{|g| > \lambda\}| = A_1 < \infty$$

Define

$$m_s = m_s(g) := \sup_{E \text{ measurable}} \frac{\left\|g\chi_E\right\|_s}{\left\|\chi_E\right\|_r} \qquad \frac{1}{r} = \frac{1}{s} - 1$$

Then $m_s \leq C_s \cdot A_1$.

Theorem 3.7 (Semmes' Tb theorem for square functions)**.** *Let* $\Theta_t f(x) = \int_{\mathbb{R}^n} \psi_t(x,y)\, f(y)\, dy$ *for* $\{\psi_t\}_{t>0}$ *in L-P class. Suppose there exists an accretive function b (i.e. b is bounded and Re* $b(x) \geq \delta$*) so that* $d\mu(x,t) = |\Theta_t b(x)|^2\, dx\, \frac{dt}{t}$ *is a Carleson measure. Then*

$$\int_0^\infty \int_{\mathbb{R}^n} |\Theta_t f(x)|^2\, dx\, \frac{dt}{t} \leq C\, \|f\|_2^2$$

Note. If $b \equiv 1$, this is Theorem 2.3. The accretive condition can be replaced with pseudo-accretivity: replace the lower bound condition with a nice approximation identity P_t with $|P_t b(x)| \geq \delta$ for all $t > 0$ and $x \in \mathbb{R}^n$.

Proof. Note by Christ-Journé theorem (Theorem 2.3), it suffices to show that $|\Theta_t 1(x)|^2\, dx\, \frac{dt}{t}$ is a Carleson measure. Fix a cube Q. Note that by approximate identity P_t associated with the pseudo-accretive condition,

$$\int_0^\infty \int_Q |\Theta_1(x)|^2\, dx\, \frac{dt}{t} \leq \frac{1}{\delta^2} \int_0^\infty \int_Q |(\Theta_t 1)\, P_t b|^2\, dx\, \frac{dt}{t}$$

Now write

$$\Theta_t 1\; P_t b = \underbrace{((\Theta_t 1) P_t - \Theta_t)}_{R_t} b + \Theta_t b$$

□

3.2 Proof of Theorem 3.4 via L-P theorem adapted to BMO

Theorem 3.8 (David/Journé/Semmes Tb theorem for SIO)**.** *Let T be a SIO with C-Z kernel. Suppose b_1 and b_2 are accrective functions so that $Tb_1, T^* b_2$ be of class BMO and $b_2 T b_1$ satisfies WBP, i.e. the operator $\widetilde{T} f = b_2 T(b_1 f)$ with kernel $b_2(x) K(x,y) b_1(y)$ satisfies WBP. Then T is L^2-bounded.*

Note. If $b_1 = b_2 \equiv 1$, then this is just T1 theorem (Theorem 2.2).

In the following, we will just prove for the case when the kernel K of T is anti-symmetric. This gives WBP. automatically, by previous result. The general result can be proven (which was done by the authors cited) roughly the same way, but here we'll avoid complicating technicalities.
As we are looking at BMO, it is reasonable to look at the action of C_γ to constant function. The following proposition shows that C_γ is constant on constants, assuming the curve is bounded.

Proposition 3.9 (Plemelj's formula)**.** *Let γ be a closed bounded Lipschitz curive, enclosing a simply connected domain $D \subseteq \mathbb{C}$. Then for a.e. $z \in \gamma$, $C_\gamma 1(z) = \frac{1}{2}$.*