Analysis Qual Problem Collection

Paul Simanjuntak

27/11/2019

Note: by default, $m(\cdot)$, $|\cdot|$, and $\lambda(\cdot)$ should denote Lebesgue measure, unless stated otherwise.

Problem 1 : U Mass. SP19 #1

(a) Give the definition of the outer/exterior measure m^* that arises in the construction of the Lebesgue measure m on $\mathbb{R}$.

(b) Prove that $m^*(A+s) = m^*(A)$ for any subset A of $\mathbb{R}$ and any $s \in \mathbb{R}$.

(c) Prove that for any non-negative Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ and for any $t \in \mathbb{R}$

$$\int_{\mathbb{R}} f(x-t)\,dm(x) = \int_{\mathbb{R}} f(x)\,dm(x)$$

(Hint: begin with simple functions).

Problem 2 : U Mass. SP19 #2

Let $f\ inL^p(\mathbb{R}^n)$ for some $p \in [1, \infty)$. For each real number $a > 0$ define

$$E_a = \{x \in \mathbb{R}^n \mid |f(x)| > a\}$$

(a) Prove that $\lim_{n\to\infty} n^p\, m(E_n) = 0$. (Hint: it may help to consider the function $n^p \chi_{E_n}$).

Problem 3 : U Mass.SP19 #3

For $x \in [0, 1]$, let

$$x = \sum_{n=1}^{\infty} \frac{b_n}{2^n} \qquad b_n \in \{0, 1\}$$

be the binary expansion of x. Let B be the set of points which admit a binary expansion with 0 in all even positions (i.e. $b_{2n} = 0$ for all n). Show that B is a set of Lebesgue measure 0.

Problem 4 : U Mass. SP19 #6

Suppose that f is non-negative and integrable on $[0, b]$, and

$$g(x) = \int_x^b \frac{f(t)}{t}\,dt \qquad \text{for } 0 < x \le b$$

Prove that g is integrable on $[0, b]$ and that

$$\int_0^b g(x)\,dx = \int_0^b f(t)\,dt$$

Problem 5 : U Mass. SP19 #7

Let $(X, \mathcal{M})$ be a measurable space; let μ be a finite, positive measure on this space. Suppose that $f_n \to f$ in measure μ. Prove that there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e. on X.

Problem 6 : U Mass. SP19 #8

Let $(X, \mathcal{M})$ be a measurable space; let μ be a finite, non-negative measure on this space; and let ν be a finite, signed measure on this space. Denote by $|\nu|$ the non-negative measure that is the total variation of ν. Prove that the following are equivalentL

(i) For all $E \in \mathcal{M}$, $|\nu|(E) \leq \mu(E)$.

(ii) $\nu \ll \mu$ and $\left|\frac{d\nu}{d\mu}(x)\right| \leq 1$ for μ-a.e. $x \in X$.

Problem 7 : U Mass. FA18 #1

Let $\{E_n\}$ be a countable collection of measurable sets in $\mathbb{R}^d$. Define

$$\limsup_{n\to\infty} E_n := \{x \in \mathbb{R}^d \mid x \in E_n, \text{ for infinitely many } n\}$$
$$\liminf_{n\to\infty} E_n := \{x \in \mathbb{R}^d \mid x \in E_n, \text{ for all but finitely many } n\}$$

(a) Show that

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \qquad \liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} E_j$$

(b) (Baby Fatou). Show that

$$m(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} m(E_n),$$

and that

$$m(\limsup_{n\to\infty} E_n) \geq \limsup_{n\to\infty} m(E_n) \text{ provided that } m\left(\bigcup_{n=1}^{\infty} E_n\right) < \infty$$

Problem 8 : U Mass. FA18 #3

Consider the function $f(x, y) := e^{-xy} - 2e^{-2xy}$ where $x \in (1, \infty)$ and $y \in (0, 1)$.

(a) Prove that for a.e. $y \in (0, 1)$, $f^y = f(\cdot, y)$ is integrable on $(1, \infty)$.

(b) Prove that for a.e. $x \in (1, \infty)$, $f^x = f(x, \cdot)$ is integrable on $(0, 1)$.

(c) Prove that $f(x, y)$ is not integrable on $(1, \infty) \times (0, 1)$.

Problem 9 : U Mass. FA18 #4

(a) Let $\{e_n\}_{n=1}^N$ be an orthonormal collection of functions in $L^2[a,b]$. Given $f \in L^2[a,b]$ find the values of $a_k \in \mathbb{R}$ which minimize $\left\| f - \sum_{n=1}^{N} a_n e_n \right\|_{L^2[a,b]}$.

(b) Suppose $\{e_n\}_{n=1}^\infty$ is an orthonormal basis for $L^2[a,b]$. Show that if $\{\varphi_n\}_{n=1}^\infty$ is another collection of functions (not necessarily orthonormal) in $L^2[a,b]$ such that

$$\sum_{n=1}^{\infty} \|e_n - \varphi_n\|_{L^2[a,b]} < 1$$

then $\{\varphi_n\}_{n=1}^\infty$ is also a complete system: that is, show if $f \in L^2[a,b]$ is orthogonal to φ_n for every $n \geq 1$ then f is the zero function.

Problem 10 : U Mass. FA18 #6

(a) Consider $f_n(x) := \chi_{[n,n+1]}(x)$ be a sequence in $L^1(\mathbb{R})$. Show that $\|f_n\|_1 = 1$ for all $n \geq 1$ and that $f_n \to 0$ pointwise but $f_n \not\to 0$ weakly in L^1.

(b) For every $n \geq 1$ let $f_n(x) := \cos(2\pi n x)$ be a sequence in $L^2[0,1)$ (verify this!). Show that $f_n \to 0$ weakly in $L^2[0,1)$ but $f_n \not\to 0$ a.e. x.

(c) Let $f_n(x) := n\chi_{\left(0,\frac{1}{n}\right)}(x)$ be a sequence in $L^2[0,1]$. Show that $f_n \to 0$ a.e. x and in measure, but $f_n \not\to 0$ weakly in $L^2[0,1]$.

Problem 11 : U Mass. FA18 #7

Let $\mathcal{H}$ be a Hilbert space, and $L : \mathcal{H} \to \mathcal{H}$ a linear function.

(a) Show that L is bounded iff L is continuous.

(b) Suppose $\|L\| < 1$, where $\|\cdot\|$ is the operator norm, and let $I : \mathcal{H} \to \mathcal{H}$ be the identity operator. Show that $I - L$ is invertible.

Problem 12 : U Mass. FA18 #8

Let $1 \leq p < \infty$ fixed. For $f \in L^p(\mathbb{R}^d)$ consider the distribution function $\lambda_f : [0,\infty] \to [0,\infty]$ defined by

$$\lambda_f(a) := m(\{x \in \mathbb{R}^d \mid |f(x)| > a\})$$

Recall that $\lambda)f$ is decreasing and right continuous and that $\int_{\mathbb{R}^d} |f(x)|^p \, dx = \int_0^\infty \alpha^{p-1} \, \lambda_f(a) \, da$.

(a) Show that if $\sum_{k=-\infty}^{\infty} 2^{kp}\lambda_f(2^k) < \infty$ then $f \in L^p(\mathbb{R}^d)$.

(b) Show that if $f \in L^p(\mathbb{R}^d)$ then $\sum_{k=-\infty}^{\infty} 2^{kp}\,\lambda_f(2^k) < \infty$. (Hint: one approach is to note that $1 = \int_0^\infty 2^{1-n}\,\chi_{[2^{n-1},2^n)}(\alpha)\,d\alpha$).

Problem 13 : U Mass. SP18 #1

Let (X, μ) be a finite measure space with $\mu(X) = 1$. The two parts of this problem are unrelated.

(a) Given $f \in L^1(X)$ define $\lambda := \int_X f\,d\mu$ and $\sigma_m = \left(\int_X |f-\lambda|^m\,d\mu\right)^{1/m}$ for $m = 1, 2, \cdots$. Prove that for any $k > 0$ and any $m = 1, 2, \cdots$,

$$\mu(\{x \in X \,|\, |f(x) - \lambda| \geq k\,\sigma^m\}) \leq k^{-m}$$

(b) Let $\{A_n\}_{n=1}^\infty$ be a sequence of measurable sets in X with $\sum_{n=1}^\infty \mu(A_n) < \infty$. Define $A = \limsup_{n\to\infty} A_n$. Clearly A is measurable. Prove that $\mu(A) = 0$.

Problem 14 : U Mass. SP18 #3

Suppose the family of measurable functions $\{K_\delta\}_{\delta>0}$ satisfies the following conditions:

(i) $\displaystyle\int_{\mathbb{R}^d} K_\delta(x)\,dx = 1$,

(ii) $\displaystyle\int_{\mathbb{R}^d} |K_\delta(x)|\,dx \leq M$, M independent of δ,

(iii) $\displaystyle\lim_{\delta\to 0}\int_{\{y\in\mathbb{R}^d \,|\, |y|\geq\eta\}} |K_\delta(x)|\,dx = 0$, for any $\eta > 0$.

Show that if f is a bounded measurable function on $\mathbb{R}^d$ which is continuous at some $x_0 \in \mathbb{R}^d$ then

$$\lim_{\delta\to 0}\int_{\mathbb{R}^d} f(x_0 - y)\,K_\delta(y)\,dy = f(x_0)$$

(Hint: Note that $f(x) = \int_{\mathbb{R}^d} K_\delta(y)\,f(x)\,dy$ for any $x \in \mathbb{R}^d$.)

Problem 15 : U Mass. SP18 #4

Suppose $f \in L^p(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for some $p > 1$.

(a) Show that $f \in L^q(\mathbb{R}^d)$ for any $q > p$. Moreover, show that $\{x \in \mathbb{R}^d \,|\, |f(x)| \geq a\}$ has finite Lebesgue measure for any $a > 0$.

(b) Show that $\lim_{q\to\infty} \|f\|_q = \|f\|_\infty$. (Hint: for any $0 < \delta < \|f\|_\infty$, consider the L^q norm of f on the set $E_\delta = \{x \in \mathbb{R}^d \,|\, |f(x)| \geq \|f\|_\infty - \delta\}$.)

Problem 16 : U Mass. SP18 #5

(a) Show that if $\{f_n\}$ converges to f in $L^p(\mathbb{R}^d)$ for some $p \geq 1$ then f_n converges to f in measure.

(b) Give an example of a sequence of functions on $\mathbb{R}$, with Lebesgue measure, which converge to zero in measure but not in $L^1(\mathbb{R})$.

(c) Give an example of sequence of functions on $\mathbb{R}$, with the Lebesgue measure, which converge to zero almost everywhere but not in measure.

(d) Give an example of a sequence of functions on $[0, 1]$, with the Lebesgue measure, which converge to zero almost everywhere and in measure, but not weakly in $L^p[0, 1]$ for any $p \geq 1$.

Problem 17 : U Mass. SP18 #7

Let $(X, \mathcal{M})$ be a measurable space.

(a) Let ν be a σ-finite measure on $(X, \mathcal{M})$. Assume that there exists a non-negative measurable function g on X having the property that for all $A \in \mathcal{M}$,

$$\nu(A) = \int_A g \, d\nu$$

Show that $g = 1$ ν-a.e. on X.

(b) Let ρ and λ be σ-finite measures on $(X, \mathcal{M})$ having the property that $\rho \ll \lambda$ and $\lambda \ll \rho$. Prove that the Radon-Nikodym derivatives satisfy

$$d\rho/d\lambda = \frac{1}{d\lambda/d\rho} \text{ a.e. with respect to either } \rho \text{ or } \lambda$$

Problem 18 : U Mass. SP18 #8

A family of functions $\{f_n\}_{n=1}^{\infty}$ on $\mathbb{R}^d$ is said to be equi-integrable if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any measurable set B with $m(B) < \delta$, then

$$\int_B |f_n| \, dm < \varepsilon \qquad \text{for all } n \geq 1$$

Show that $\{f_n\}$ is equi-integrable and A is a measurable set with $m(A) < \infty$ and $f_n \to f$ a.e. in A, then

$$\lim_{n\to\infty} \int_A f_n \, dm = \int_A f \, dm$$

(Hint: Use Egoroff and Fatou.)

Problem 19 : U Mass. FA17 #1

(a) Let $\mathcal{A}$ be a σ-algebra on a set X and let $\{B_k\}$ be a sequence of pairwise disjoint sets. For each $n \in \mathbb{N}$ define $A_n := \bigcup_{k=n}^{\infty} B_k$. Prove that $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

(b) Let $\mathcal{A}$ be a σ-algebra on a set X and assume that $\mu : \mathcal{A} \to [0, \infty]$ has the following properties:

 (i) If $A_1, A_2 \in \mathcal{A}$ with $A_1 \cap A_2 = \emptyset$, then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$.

 (ii) If $\{A_n\}$ is a sequence in $\mathcal{A}$ such that $A_{n+1} \subseteq A_n$ for all n, and $\bigcap_{n=1}^{\infty} A_n = \emptyset$, then $\lim_{n\to\infty} \mu(A_n) = 0$.

Prove that μ is a positive measure on X.

Problem 20 : U Mass. FA17 #2

(a) Compute the following limit justifying all steps,

$$\lim_{n\to\infty} \int_0^{\infty} \frac{n \sin(x/n)}{x(1+x^2)}\, dx$$

(b) Suppose $g : \mathbb{R} \to [0, \infty)$ is in $L^1(\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} n\, g(nx)\, f(x)\, dx = f(0)\ \|g\|_1$$

Problem 21 : U Mass. FA17 #3

(a) Suppose that $\{f_n\}, \{g_n\}, f, g$ are functions in $L^1(\mathbb{R}^d)$ and that $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for a.e. $x \in \mathbb{R}^d$ as $n \to \infty$. Show that if $|f_n| \le g_n$ and $\lim_{n\to\infty} \int g_n = \int g$ then $\lim_{n\to\infty} \int f_n = \int f$.

(b) Suppose $\{f_n\}, f$ are functions in $L^1(\mathbb{R}^d)$ and that $f_n(x) \to f(x)$ for a.e. $x \in \mathbb{R}^d$. Prove that $f_n \to f$ in $L^1(\mathbb{R}^d)$ iff $\|f_n\|_1 \to \|f\|_1$.

Problem 22 : U Mass. FA17 #4

(a) Let (X, μ) be a measure space and let $f \in L^1(X)$. Prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \int_A f\, d\mu \right| < \varepsilon$$

whenever A is a measurable subset of X with $\mu(A) < \delta$.

(b) Consider $(\mathbb{R}, m)$. Suppose that λ is a finite positive measure on $\mathbb{R}$ which is absolutely continuous with respect to m. Prove that the function $F(x) := \lambda((-\infty, x])$, $x \in \mathbb{R}$, is uniformly continuous.

Problem 23 : U Mass. FA17 #5

Let N be a fixed positive integer and for any $\sigma > 0$ consider the function

$$\psi_\sigma(x) := \begin{cases} (s/\sigma)^N & \text{if } 0 < s \leq \sigma \\ 0 & \text{if } \sigma < s < \infty \end{cases}$$

(a) Show that for any $a > 0$, $\psi_{a\sigma}(s) = \psi_\sigma(a^{-1}s)$.

(b) Suppose that g is a non-negative function in $L^1\left([0,\infty), \frac{dx}{x}\right)$. Show that

$$\int_0^\infty \int_s^{2s} \psi_\sigma(s)\, g(t)\, \frac{dt}{t}\, \frac{ds}{s} = \int_\sigma^{2\sigma} \int_0^\infty \psi_u(s)\, g(s)\, \frac{ds}{s}\, \frac{du}{u}$$

Problem 24 : U Mass. FA17 #7

(a) Let f be an integrable function defined on $[a,b]$ and let ϕ be a continuous convex function defined on $\mathbb{R}$. Prove that

$$\phi\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \leq \frac{1}{b-a}\int_a^b \phi(f(x))\,dx$$

(b) Show that if $f \in L^q[0,1]$. $q > 0$, then

$$\int_0^1 \log|f(x)|\,dx \leq \log\|f\|_q$$

Problem 25 : U Mass. FA17 #8

(a) Let $1 < p < q < \infty$. Show that $L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ with the norm $\|f\|_{L^p \cap L^q} = \|f\|_p + \|f\|_q$.

(b) Let $1 < p < r < q < \infty$. Show that $L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \subseteq L^r(\mathbb{R}^d)$ and that the inclusion map is continuous with respect to the norm in part (a).

Problem 26 : Wisconsin 8/2018 #4

(a) Construct a Lebesgue measureble subset $E \subseteq \mathbb{R}$ such that $0 < |E \cap I| < |I|$ for every non-empty finite interval I.

(b) Can there exist a constant $\alpha > 0$ and a measurable set E as above such that E further satisfies $|E \cap I| > \alpha |I|$ for every non-empty finite interval I?

Problem 27 : U Mass. SP17#1
Let f_n and g_n be sequences in $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$, respectively, such that $1/p + 1/q = 1$ and for some $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ we have

$$f_n \to f \text{ in } L^p \qquad g_n \to g \text{ in } L^q$$

Show that the sequence $h_n := f_n g_n$ converges in L^1 to $h := fg$.

Problem 28 : U Mass. SP17 #2
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function with compact support. Show that

$$\lim_{n\to\infty} \int_{\{(x,y)\,|\,x\geq 0, y\geq 0\}} \frac{n}{2\pi} f(x,y) \exp\left(-n\frac{x^2}{2} - n\frac{y^2}{2}\right) dx\,dy = \frac{1}{4} f(0,0)$$

(Hint: think about what the limit should be if you changed the area of integration from the upper right quadrant to all of $\mathbb{R}^2$. You may recall the fact $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$.)

Problem 29 : U Mass. SP17 #3
Fix f such that $f \in L^1(0,\infty) \cap L^2(0,\infty)$.

(a) Show that $f \in L^r(0,\infty)$ for every $r \in (1,2)$.

(b) Show that the function

$$\phi(r) = \|f\|_r \qquad r \in [1,2]$$

is a continuous function of r.

(Hint: show first that $g(x) = \max\{f(x), f(x)^2\}$ is an integrable function in $(0,\infty)$, or use Hölder inequality for properly chosen exponents.

Problem 30 : U Mass. SP17 #5
Consider a sequence of measurable functions $g_n(x)$ on $\mathbb{R}$ such that $\lim_{n\to\infty} g_n(x)$ exists for almost every $x \in \mathbb{R}$. Suppose further that for some $p \in [1,\infty)$ we have

$$G(x) := \sup_n |g_n(x)|^p \in L^1(\mathbb{R})$$

Then prove that $g_n \to g$ in the $L^p(\mathbb{R})$ norm.

Problem 31 : U Mass. SP17 #6
Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be a measurable function such that

$$\mu(\{n \leq |f| \leq n+1\}) \leq \frac{M}{2^n}$$

Show that for any $0 < p < \infty$, we have

$$\int_X |f|^p\,d\mu < \infty$$

Problem 32 : U Mass. SP17 #7

Consider a set X with a σ-algebra $\mathcal{M}$ and a family m_n of measures with respect to $\mathcal{M}$. Suppose that

$$\sup_n m_n(X) < \infty$$

Define a new measure m by setting, for each $E \in \mathcal{M}$,

$$m(E) := \sum_{n=1}^{\infty} \frac{1}{n^2} m_n(E)$$

Show m is also a measure in $\mathcal{M}$ and that each m_n is absolutely continuous with respect to m.

Problem 33 : U Mass. SP17 #8

Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is locally integrable (i.e. f is integrable on any compact set of $\mathbb{R}$). Define the maximal function of f by

$$f^*(x) = \sup_{I \ni x} \frac{1}{I} \int_I |f(y)|\, dy$$

Here the supremum is taken over all intervals containing x. Suppose that f is not almost everywhere equal to zero.

(a) Show that if $f \in L^\infty(\mathbb{R})$, then $\|f^*\|_\infty \leq \|f\|_\infty$.

(b) Show there exists some $c > 0$ such that

$$f^*(x) \geq \frac{c}{|x|} \text{ whenever } |x| \geq 1$$

(c) Conclude that the maximal function of $f \in L^1_{\text{loc}}(\mathbb{R})$ (locally integrable functions) belongs to $L^1(\mathbb{R})$ iff f is zero a.e.

Problem 34 : U Mass. FA16 #1

Let $f : [0,1] \in \mathbb{C}$ be a continuous function. Prove that

$$\lim_{n\to\infty} \int_0^1 e^{-2\pi n x} f(x)\, dx = 0$$

Either by proof or example, determine whether there can be an f as above, and such that

$$\left| \int_0^1 e^{-2\pi n x} f(x)\, dx \right|^2 \geq \frac{1}{n} \qquad \forall n \in \mathbb{N}$$

(Hint: consider the Hilbert space $L^2(0,1)$ with the usual product.)

Problem 35 : U Mass. FA16 #4
Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable, non-negative function. Define $\lambda_f : (0, \infty) \to \mathbb{R}$ by

$$\lambda_f(\alpha) = m(\{x \in \mathbb{R}^d \mid f(x) > \alpha\})$$

Prove that λ_f is Lebesgue measurable and

$$\int_{\mathbb{R}^d} f(x)\, dx = \int_0^\infty \lambda_f(\alpha)\, d\alpha$$

Problem 36 : U Mass. FA16 #5
Let μ be a finite measure on the Borel set $X = [0, 1]$ such that $\mu(\{x\}) = 0$ for all $x \in X$. Prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mu(A) \le \varepsilon$ for all intervals $A \subseteq X$ contained in $\left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right)$.

Problem 37 : U Mass. FA16 #6
Given $f : \mathbb{R} \to \mathbb{R}$ define $f_h(x) = f(x - h)$.

(a) Show that if f is continuous with compact support then $\lim_{h\to 0} \|f_h - f\|_\infty = 0$.

(b) Show that if $f \in L^p(\mathbb{R})$ with $1 \le p < \infty$ then $\lim_{h\to 0} \|f_h - f\|_p = 0$.

(c) Prove or disprove by a counter-example: if $f \in L^\infty(\mathbb{R}^n)$, then $\lim_{h\to 0} \|f_h - f\|_\infty = 0$.

Problem 38 : U Mass. FA16 #8
Consider Lebesgue measure on Borel sets of $(0, \infty)$. Prove that for every $f \in L^2(0, \infty)$

(a) The inequality

$$\left|\int_0^x f(x)\, dx\right|^2 \le 2\sqrt{x} \int_0^x \sqrt{s}\, |f(s)|^2\, ds$$

holds for all $x \in (0, \infty)$.

(b) The inequality $\|F\|_2 \le 2\, \|f\|_2$, where $F(x) = \dfrac{1}{x} \int_0^x f(s)\, ds$.

Problem 39 : U Mass. SP16 #1
Let $(X, \mathcal{M}, \mu)$ be a finite measure space with $\mu(X) > 0$ and let f be a non-negative, Borel-measurable function mapping X into $[0, \infty)$. Assume that $f \neq 0$ a.e. For $A \in \mathcal{M}$, define

$$\nu(A) = \int_A f\, d\mu$$

(a) Prove that there exists $A \in \mathcal{M}$ such that $\nu(A) > 0$. This proves that ν is non-trivial.

(b) Prove that ν is a measure on $\mathcal{M}$ and that $\nu \ll \mu$.

(c) Let ψ be any non-negative, Borel-measurable function function mapping X into $[0, \infty)$. Prove that

$$\int_X \psi \, d\nu = \int_X \psi \, f \, d\mu$$

(d) In the particular case where f is also a bounded function, prove that ν is a finite measure.

Problem 40 : U Mass. SP16 #8

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. For $\alpha \in \mathbb{R}$, define $\lambda(\alpha) = \mu(\{|f| > \alpha\})$.

(a) Prove that for $f \in L^1(\mu)$,

$$\int_X |f| \, d\mu = \int_0^\infty \lambda(\alpha) \, d\alpha$$

(b) Prove that for $f \in L^p(\mu)$ for $1 < p < \infty$

$$\int_X |f|^p \, d\mu = p \int_0^\infty \alpha^{p-1} \, \lambda(\alpha) \, d\alpha$$

Problem 41 : U Mass. FA15 #1

Consider the space $[0, 1]$ with Lebesgue measure. For $n \in \mathbb{N}$, define the intervals $A_n = \left(\frac{1}{n+1}, \frac{1}{n}\right]$. For $\alpha \in \mathbb{R}$ define the function $f : [0, 1] \to \mathbb{R}$ by $f(0) = 0$ and

$$f(x) = \sum_{n=1}^{\infty} n^\alpha \, \chi_{A_n}(x)$$

Find the values of $\alpha \in \mathbb{R}$ for which f is integrable. Then prove that f is integrable for these values of $\alpha \in \mathbb{R}$.

Problem 42 : U Mass. FA15 #2

Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of real-valued functions in $L^2(X)$ such that

$$\sup_n \int_X f_n^2 \, d\mu \leq C < \infty$$

for some positive constant C. Prove that $\lim_{n\to\infty} \dfrac{f_n(x)}{n} = 0$ for μ-a.e. $x \in X$. (Hint: consider the sequence of functions $g_n = \sum_{k=1}^n f_k^2/k^2$ for $n \in \mathbb{N}$.)

Problem 43 : U Mass. FA15 #6

Let E be a non-empty closed and convex set in a Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$. Prove that there exists an unique element $x_0 \in E$ which minimizes $\|x\|$ on E, i.e. $\|x_0\| = \inf_{x \in E} \|x\|$.

Problem 44 : U Mass. FA15 #8

(a) Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) < \infty$. Let $\{f_n\}$ and f be measurable functions that map X into $\mathbb{R}$. Prove that if $f_n \to f$ μ-a.e. then $f_n \to f$ in measure. (Hint: one way to prove this is Egoroff's Theorem.)

(b) Show that Egoroff's theorem fails for the measure space $(\mathbb{R}, \mathcal{B}, m)$.

Problem 45 : U Mass. FA14 #2

For $x \in (0, \infty)$ consider the integral

$$F(x) = \int_0^\infty \frac{1 - \exp(-xt^2)}{t^2}\, dt$$

(a) Prove that $F(x) < \infty$ for all $x \in (0, \infty)$.

(b) By calculating the derivative $F'(x)$ and using the fact that $\int_{-\infty}^{\infty} e^{-s^2}\, ds = \sqrt{\pi}$, compute $F(x)$ as an explicit function of x.

Problem 46 : U Mass. FA14 #4

(a) Let $\{f_n\}$ be a sequence of real-valued, measurable functions on $[0, 1]$. Assume that for each $n \in \mathbb{N}$

$$\int_0^1 f_n^2\, dm \le \frac{1}{n^3}$$

Prove that $f_n \to 0$ a.e. on $[0, 1]$ as $n \to \infty$.

(b) Let $\{g_n\}$ be a sequence of real-valued measurable functions on $[0, 1]$. Assume that for each $n \in \mathbb{N}$

$$\int_0^1 g_n^2\, dm \to 0 \text{ as } n \to \infty$$

By giving a counter-example, prove that it does not follow that $g_n \to 0$ a.e. on $[0, 1]$.

Problem 47 : U Mass. FA14 #5

Let $(X, \mathcal{M}, \mu)$ be a measure space, and fix finite real numbers p and q satisfying $1/p + 1/q = 1$. Let $\{f_n\}$ be a sequence of functions in $L^p(X)$ converging in $L^p(X)$ to f, and let $\{g_n\}$ be a sequence of functions in $L^q(X)$ converging in $L^q(X)$ to q. Prove that the sequence $\{f_n g_n\}$ converges in $L^1(X)$ to fg.

Problem 48 : U Mass. FA14 #7

FOr $1 \le p \le \infty$, let ℓ^p denote the normed vector space of real sequences $\{x_n\}$ that equal $L^p(\mathbb{N})$ for counting measure on $\mathbb{N}$. Prove or disprove the following statements:

(a) ℓ^1 is separable.

(b) ℓ^∞ is separable.

Problem 49 : U Mass. SP14 #1
Let f be a bounded continuous function mapping $[0, \infty)$ into $\mathbb{R}$. Prove that

$$\lim_{n\to\infty} \int_0^\infty n\, e^{-nx} f(x)\, dx = f(0)$$

Problem 50 : U Mass. SP14 #3
All the spaces $L^p[0,1]$ in this problem are defined with respect to the Lebesgue measure.

(a) Prove that $L^4[0,1] \subseteq L^3[0,1]$.

(b) Assume that $\Lambda : L^3[0,1] \to \mathbb{R}$ is a bounded linear functional Prove that the restriction of Λ to $L^4[0,1]$ is a bounded linear functional on $L^4[0,1]$.

(c) Give an example (with proof) of a function in $L^{4/3}[0,1]$ that is not in $L^3[0,1]$.

(d) Give an example (with proof) of a bounded linear functional on $L^3[0,1]$ that is not the restriction to $L^3[0,1]$ of a bounded linear functional on $L^2[0,1]$.

Problem 51 : U Mass. SP14 #5
Consider the space $C[0,1]$ normed with two norms:

$$\|f\|_1 = \int_0^1 |f(x)|\, dx \qquad \|f\|_2 = \sup_{x\in[0,1]} |f(x)|$$

Also let $I : C[0,1] \to C[0,1]$ be the identity map $I(f) = f$.

(a) Is the mapping $I : (C[0,1], \|\cdot\|_1) \to (C[0,1], \|\cdot\|_2)$ continuous? Give proof or counter-example.

(b) Is the mapping $I : (C[0,1], \|\cdot\|_2) \to (C[0,1], \|\cdot\|_1)$ continuous? Give proof or counter-example.

(c) The space $C[0,1]$ is complete under either $\|\cdot\|_1$ or $\|\cdot\|_2$, but not both. Which is which? Explain your answer, but you don't have to provide detailed proof.

Problem 52 : U Mass. SP14 #6
Consider the real Hilbert space $L^2[-1,1]$ defined with respect to the Lebesgue measure.

(a) Determine an orthonormal set $\{\varphi_0, \varphi_1, \varphi_2\}$ in $L^2[-1,1]$ such that the linear span of $\{1, x, x^2\}$ coincides with the linear span of $\{\varphi_0, \varphi_1, \varphi_2\}$.

(b) Compute

$$\min_{a,b\in\mathbb{R}} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2\,dx$$

(c) Compute

$$\max \int_{-1}^{1} x^3\,g(x)\,dx$$

where $g \in L^2[-1,1]$ is subject to the restriction

$$\int_{-1}^{1} g(x)\,dx = \int_{-1}^{1} x\,g(x)\,dx = \int_{-1}^{1} x^2\,g(x)\,dx = 0 \text{ and } \int_{-1}^{1} |g(x)|^2\,dx = 1$$

Problem 53 : U Mass. SP14 #8

Let $(X, \mathcal{M})$ be a measurable space and let $\{\mu_n\}$ be a sequence of uniformly bounded measures on $\mathcal{M}$. Prove that

$$\mu = \sum_{n=1}^{\infty} \frac{\mu_n}{2^n}$$

is a measure on $\mathcal{M}$ and that each μ_n is absolutely continuous with respect to μ.

Problem 54 : U Mass. FA13 #3

Let $C_{per}(\mathbb{R})$ denotes the Banach space of bounded, continuous, real-valued functions on $\mathbb{R}$ which are periodic of period 2 with the norm $\|f\| = \sup_{|x|<1} |f(x)|$.
For $n \in \mathbb{N}$ let k_n be a non-negative function in $C_{per}(\mathbb{R})$ and for $g \in C_{per}(\mathbb{R})$ define

$$S_n g(x) := \int_{-1}^{1} k_n(y)\,g(x+y)\,dy$$

(a) Prove that S_n defines a bounded linear operator from $C_{per}(\mathbb{R})$ into $C_{per}(\mathbb{R})$.

(b) Assume that for every $n \in \mathbb{N}$ we have $\int_{-1}^{1} k_n(y)\,dy = 1$ and that for each $\delta > 0$

$$\lim_{n\to\infty} \sup_{\delta\le|y|\le 1} k_n(y) = 0$$

Prove that

$$\lim_{n\to\infty} \|S_n - I\| = 0$$

where I is the identity operator on $C_{per}(\mathbb{R})$.

Problem 55 : U Mass. FA13 #7

Let $(X, \mathcal{M}, \mu)$ be a measure space, and for $1 \le p \le \infty$ consider the Banach space $L^p(X,\mu)$ and $L^q(X,\mu)$, where q is the conjugate exponent to p.

(a) Prove that for $1 < p < \infty$ and $f \in L^p(X, \mu)$ we have

$$\|f\|_p = \sup_{\|g\|_q = 1} \left| \int f \, g \, d\mu \right|$$

(b) Let $(X_1, \mathcal{M}_1, \mu_1)$ and $(X_2, \mathcal{M}_2, \mu_2)$ be two σ-finite measure spaces, and let $f(x_1, x_2)$ be a measurable, non-negative function on $X_1 \times X_2$. Prove that for $1 < p < \infty$

$$\left\| \int f(x_1, x_2) \, d\mu_2(x_2) \right\|_{L^p(X_1)} \leq \int \left\| f(x_1, x_2) \right\|_{L^p(X_1)} \, d\mu_2(x_2)$$

Problem 56 : U Mass. FA13 #8

For $-\infty < a < b < \infty$ let $f_n : (a, b) \to \mathbb{R}$ be a sequence of functions, each of which is a monotonically increasing function. Suppose that f_n converges to some f almost everywhere with respect to Lebesgue measure. Show that f_n converges to f at all points where f is continuous. (Hint: approximate any point of continuity of f by sequences lying in the set $\{f_n \to f\}$.)

Problem 57 : U Mass. SP13 #6

Let $1 \leq p < q < r \leq \infty$.

(a) Show that $L^q \subseteq L^p + L^r$, i.e. any $f \in L^q$ can be written as $f = g + h$ where $g \in L^p$ and $h \in L^r$.

(b) Show that $L^p \cap L^r \subseteq L^q$ and that for any $f \in L^p \cap L^r$, $\|f\|_q \leq \|f\|_p^{\lambda} \|f\|_r^{1-\lambda}$ for a suitable λ.

Problem 58 : U Mass. SP13 #7

Let $(X, \|\cdot\|)$ be a Banach space. Suppose that X can be written as the direct sum of two linear subspaces M and N, i.e. $x \in X$ can be expressed uniquely as $x = y + z$ where $y \in M$ and $z \in N$.

(a) For any $x \in X$ define $\|x\|' = \|y\| + \|z\|$. Prove that $\|\cdot\|'$ defines a norm on X.

(b) Consider the normed vector space $(X, \|\cdot\|')$. Which additional property of the subspaces M and N is needed to ensure that $(X, \|\cdot\|')$ is a Banach space? Prove your answer.

Problem 59 : U Mass. SP13 #8

(a) (Repeat of Problem 34) Suppose that $f \in L^1[0, 1]$ and define for $n \in \mathbb{Z}$

$$c_n = \int_0^1 f(x) \, e^{-2\pi i n x} \, dx$$

Prove the Riemann-Lebesgue lemma, i.e.

$$\lim_{n \to \infty} c_n = 0$$

(b) Suppose that E is a measurable subset of $[0, 1]$ and $\{u_n\}$ an arbitrary sequence of real numbers. Prove that

$$\lim_{n\to\infty} \int_E \cos^2(nx + u_n)\, dx = \frac{m(E)}{2}$$

$\left(\text{Hint: } \cos^2 \alpha = \frac{2}{1}2(1 + \cos(2\alpha)),\ \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\right)$

Problem 60 : U Mass FA12 #6

Let $(X, \mathcal{M}, \mu)$ be a measure space with μ a σ-finite measure and let $f \in L^1(\mu)$ be a non-negative function.

(a) Show that

$$(\mu \times m)\{(x, y)\, inX \times \mathbb{R} \mid 0 \leq f(x) \leq y\} = \int_X f\, d\mu$$

i.e. the integral of f equals the area under the graph.

(b) Show that

$$(\mu \times m)\{(x, y) \in X \times \mathbb{R} \mid f(x) = y\} = 0$$

i.e. the measure of the graph of f equals 0.

Problem 61 : U Mass. FA12 #8

Suppose that $\mathcal{H}$ is an infinite-dimensional Hilbert space.

(a) Show that there exists a sequence $\{f_n\}$ with $\|f_n\| = 1$ such that $\{f_n\}$ has no convergent subsequence.

(b) Show that for any sequence $\{f_n\}$ with $\|f_n\| = 1$ there exists a subsequence $\{f_{n_k}\}$ and $f \in \mathcal{H}$ such that

$$\lim_{k\to\infty} \langle f_{n_k}, g\rangle = \langle f, g\rangle$$

for all $g \in \mathcal{H}$. Here $\langle\cdot,\cdot\rangle$ is the inner product on $\mathcal{H}$. (Hint: use diagonalization argument.)

Problem 62 : U Mass. FA12 #5

(Lax-Milgram lemma) Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle x, y\rangle$ and let $B : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ be a bilinear functional, i.e. $B(x, y)$ is linear in x and y separately. Assume there exist constants $K > 0$ and $d > 0$ such that B satisfies the following inequalities.

$$|B(x, y)| \leq K\, \|x\|\, \|y\| \text{ for all } x, y \in \mathcal{H}$$
$$B(x, x) \geq d\, \|x\|^2 \text{ for all } x \in \mathcal{H}$$

(a) SHow that for each $z \in \mathcal{H}$ there is an uniquely determined $y \in \mathcal{H}$ such that $\langle y, x, =\rangle B(z, x)$ for all $x \in \mathcal{H}$.

(b) Prove that if the correspondence in part (a) is denoted by $y = Az$, then A is a bounded linear operator on $\mathcal{H}$ that is injective and which has closed range, i.e. the subspace $\mathcal{R} = A(\mathcal{H})$ is a closed subspace of $\mathcal{H}$.

(c) Prove that $\mathcal{R} = \mathcal{H}$, and that for any bounded lienar functional F on $\mathcal{H}$ there exists an unique $z \in \mathcal{H}$ so that $F(x) = B(z, x)$ for all $x \in \mathcal{H}$.

Problem 63 : U Mass. SP12 #6

(a) For any given $1 \leq p < \infty$, find an example of an unbounded continuous function in $L^p(\mathbb{R})$.

(b) Show that if $1 \leq p < \infty$ and $f \in L^p(\mathbb{R})$ is a function that is uniformly continuous on $\mathbb{R}$, then $\lim_{x \to \pm\infty} f(x) = 0$ (i.e. f tends to 0 away from $x = 0$).

Problem 64 : U Mass. SP12 #7

Let X be a Banach space and let $T : X \to X$ be a bounded linear operator with $\|T\| < 1$, where

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$$

Prove that the operator $I - T$ has a bounded inverse, and that

$$\left\|(I - T)^{-1}\right\| \leq \frac{1}{1 - \|T\|}$$

Problem 65 : Wisconsin 8/2018 #5

(a) Give an example, with explanation, of each of the following:

- A sequence of functions on $\mathbb{R}$ that converges to zero in $L^1(\mathbb{R})$, but it does not converge almost everywhere on $\mathbb{R}$ to any function.
- A sequence of functions in $L^1(\mathbb{R})$ that converges almost everywhere to zero, but it does not converge in measure to any function

(b) Prove that a sequence of functions on $\mathbb{R}$ that converges to zero in measure must have a subsequence that converges to zero almost everywhere. Do not quote any theorems that trivialize this problem.

Problem 66 : Wisconsin 8/2018 #6

Prove that in an infinite dimensional Banach space,

(a) every norm bounded set is weakly bounded,

(b) every norm closed set is weakly closed,

(c) a norm-bounded set has empty interior in the weak topology.

Problem 67 : Wisconsin 8/2018 #8R

Let $1 < p \leq \infty$. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $\{f_n\}$ be a sequence of measurable functions converging μ-a.e. to the function f. Assume further that $\|f_n\|_p \leq 1$ for all n. Prove that $f_n \to f$ as $n \to \infty$ in L^r for all $1 \leq r < p$.

Problem 68 : Wisconsin 1/2018 #4

(a) Construct a Lebesgue measurable subset $E \subseteq \mathbb{R}$ such that for every finite, non-empty open interval I, $0 < \frac{|E \cap I|}{|I|} < 1$.

(b) Prove or disprove that there exists a Lebesgue measurable set $E \subseteq \mathbb{R}$ and some $0 < \theta < 1$ such that $\frac{|E \cap I|}{|I|} = \theta$, for every finite, non-empty open interval $I \subseteq \mathbb{R}$.

Problem 69 : Wisconsin 1/2018 #5

Let $\{x_n\}$ be a sequence in a Hilbert space H. Suppose that x_n converges to x weakly. Prove that there is a subsequence x_{n_k} such that

$$\frac{1}{N} \sum_{k=1}^{N} x_{n_k}$$

converges to x in norm as $N \to \infty$.

Problem 70 : Wisconsin 1/2018 #6

For $f \in L^1(\mathbb{R})$, define

$$T_\varepsilon f(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\varepsilon}{y^2 + \varepsilon^2} f(x - y)\, dy$$

Prove that $T_\varepsilon f(x) \to f(x)$, as $\varepsilon \to 0^+$, for a.e. x.

Problem 71 : Wisconsin 1/2018 #9R

Let X and Y be Banach spaces, and let $\{T_n\} \subseteq \mathcal{L}(X, Y)$, where $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X to Y.

(a) Prove that if $T_n \to T$ strongly (i.e. pointwise) and $x_n \to x$ (i.e. in norm), then $T_n x_n \to T x$.

(b) If $T_n \to T$ strongly and $x_n \to x$ weakly, must $T_n x_n \to T x$ weakly?

Problem 72 : Wisconsin 8/2017 #5

Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported function that satisfies the Hölder condition with exponent $\beta \in (0, 1)$, i.e. there exists a constant $A < \infty$ such that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq A|x - y|^\beta$. Consider the function g defined by

$$g(x) = \int_{-\infty}^{\infty} \frac{f(y)}{|x - y|^\alpha}\, dy$$

where $\alpha \in (0, \beta)$.

(a) Prove that g is a continuous function at 0.

(b) Prove that g is differentiable at 0 (Hint: Try the Dominated Convergence Theorem).

Problem 73 : Wisconsin 8/2017 #9

Let f_n be a sequence of real functions on $\mathbb{R}$ such that each f_n' is continuous on $\mathbb{R}$. Suppose that as $n \to \infty$, f_n converges to a function $f : \mathbb{R} \to \mathbb{R}$ pointwise and f_n' converges to a function $g : \mathbb{R} \to \mathbb{R}$ pointwise.
Prove that there exists a non-empty interval (a, b) and a constant $L < \infty$ such that

$$|f(x) - f(y)| \le L|x - y|, \qquad \forall\, x, y \in (a, b)$$

(Hint: Consider the sets $K_c = \{x \in \mathbb{R} \mid \forall n, |f_n'(x)| \le c\}$.)

Problem 74 : Wisconsin 1/2017 #5

Let $P(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{N} a_{mn} x^m y^n$ be a real-valued polynomial on $\mathbb{R}^2$ which is not identically zero. Prove that the set

$$\{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}$$

has Lebesgue measure zero.

Problem 75 : Wisconsin 1/2017 #6

Let $E \subseteq \mathbb{R}^n$ be a set of finite, positive measure, and let $\{t_k\}$ be a sequence with $t_k > 0$ and $\lim_{k\to\infty} t_k = 0$. Define for $f \in L^p(\mathbb{R}^n)$,

$$Mf(x) = \sup_k \frac{1}{t_k^n |E|} \int_{t_k E} |f(x - y)|\, dy$$

where $|\cdot|$ is the Lebesgue measure and the integral is taken against this measure. Suppose furthermore that there is $C > 0$ such that for every $f \in L^p(\mathbb{R}^n)$ and every $\lambda > 0$ the inequality

$$|\{x \in \mathbb{R}^n \mid Mf(x) > \lambda\}| \le C\, \lambda^{-p}\, \|f\|_p^p$$

holds.
Show that for every $f \in L^p(\mathbb{R}^n)$,

$$\lim_{k\to\infty} \frac{1}{t_k^n |E|} \int_{t_k E} f(x - y)\, dy = f(x)$$

for almost every $x \in \mathbb{R}^n$.

Problem 76 : Wisconsin 8/2016 #4

Let $f : [0, 1] \to \mathbb{R}$ be continuous with

$$\min_{0<x<1} f(x) = 0$$

Assume that for all $0 \le a \le b \le 1$ we have

$$\int_a^b [f(x) - \min_{a\le y\le b} f(y)]\,dx \le \frac{|b-a|}{2}$$

Prove that for all $\lambda \ge 0$ we have

$$|\{x \in [0,1] \mid f(x) > \lambda + 1\}| \le \frac{1}{2}|\{x \in [0,1] \mid f(x) > \lambda\}$$

Problem 77 : Wisconsin 8/2016 #5
Give an example of a non-empty closed subset of $L^2[0,1]$ that does not contain a vector of smallest norm. Prove your assertion.

Problem 78 : Wisconsin 8/2016
Let f be a non-negative measurable function on $[0,1]$. Prove that the following statements are equivalent:

(a) There exists $a > 0$ such that

$$\int_0^1 e^{a f(x)}\,dx < \infty$$

(b) There exists $C > 0$ such that

$$\left(\int_0^1 f(x)^p\,dx\right)^{1/p} \le Cp \text{ for all} 1 \le p < \infty$$

Problem 79 : Wisconsin 8/2016 #8
Show that there is a continuous real-valued function on $[0,1]$ that is not monotone on any open interval $(a,b) \subseteq [0,1]$.

Problem 80 : Wisconsin 1/2016 #4
Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set with $|E| < \infty$. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(r) = |E \cap (E + r)|$ is continuous.

Problem 81 : Wisconsin 1/2016 #5
For $f \in L^1(\mathbb{R}^2)$ let

$$T_n f(x) = \iint_{\mathbb{R}^2} n^4 e^{-n\sqrt{|n_1|+|n_2|}} f(x_1 - y_1, x_2 - y_2)\,dy_1\,dy_2$$

Prove that for almost every $x \in \mathbb{R}^2$, $T_n f(x)$ converges as $n \to \infty$.

Problem 82 : Wisconsin 1/2016 #6
Let $1 < p < \infty$, and let $\chi_{[n,n+1]}$ denote the characteristic function of $[n, n+1]$. For which $\alpha \in \mathbb{R}$ does the sequence $n^\alpha \chi_{[n,n+1]}$ converge weakly to 0 in $L^p(\mathbb{R})$?

Problem 83 : Wisconsin 1/2016 #7
For simple functions on $[-1, 1]$, define

$$Tf(x) = \int_{-1}^{1} \frac{2^{x-y}}{|x| + |y|} f(y)\, dy$$

Show that T extends to an operator which maps $L^2[-1, 1]$ to $L^2[-1, 1]$.

Problem 84 : Wisconsin 8/2015 #4
Let $E \subseteq \mathbb{R}$ be measurable and satisfies

$$E + r = E$$

for every rational number r. Show that either E or its complement has measure 0.

Problem 85 : Wisconsin 8/2015 #6
Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be measurable. Prove that if $1 \le p < r < q < \infty$ and there is $C < \infty$ such that

$$\mu(\{x \,|\, |f(x)| > \lambda\}) \le \frac{C}{\lambda^p + \lambda^q}$$

for every $\lambda > 0$, then $f \in L^r(\mu)$.

Problem 86 : Wisconsin 8/2015 #7
Let $p \in (1, \infty)$, and for $f \in L^p(\mathbb{R})$ define

$$Tf(x) = \int_0^1 f(x+y)\, dy$$

(a) Show that $\|Tf\|_p \le \|f\|_p$, and equality holds iff $f = 0$ almost everywhere.

(b) Prove that the map $f \mapsto f - Tf$ does not map $L^p(\mathbb{R})$ onto $L^p(\mathbb{R})$.

Problem 87 : Wisconsin 1/2015 #4
Let $r_n \in [0, 1]$ be an arbitrary sequence and define the function

$$f(x) = \sum_{r_n < x} \frac{1}{2^n}$$

Show that f is Borel measurable, find all its points of discontinuity, and find $\displaystyle\int_0^1 f(x)\, dx$.

Problem 88 : Wisconsin 1/2015 #5

Let $f \in L^2[0,1]$ satisfies $\int_0^1 t^n f(t)\,dt = (n+2)^{-1}$ for $n = 0, 1, 2, \cdots$. Must then $f(t) = t$ a.e.?

Problem 89 : Wisconsin 1/2015 #6

Does there exist a Borel measurable function $f : \mathbb{R} \to [0, \infty)$ such that $\int_a^b f(x)\,dx = \infty$ for all real numbers $a < b$? Find an example or show that no such f exists.

Problem 90 : Wisconsin 1/2015 #8

Let $\alpha \in (0,1)$, and for $f \in C[0,1]$ and $x \in [0,1]$ define

$$(T_\alpha f)(x) = \int_0^1 \frac{\sin(x+y)}{|x-y|^\alpha} f(y)\,dy$$

(i) Prove that T_α extends to a bounded linear operator on $L^2[0,1]$.

(ii) For which $\alpha \in (0,1)$ is $T_\alpha : L^2[0,1] \to L^2[0,1]$ a compact operator?

Problem 91 : Wisconsin 8/2014 #4

Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $f_n \to f$ in measure and $g_n \to g$ in measure.

(a) Show that $f_n g_n \to fg$ in measure if $\mu(X) < \infty$.

(b) What if $\mu(X) = \infty$? Either prove the result in this case or construct a counter-example.

Problem 92 : Wisconsin 8/2014 #6

Let X and Y be Banach spaces and $\{T_{j,k} \mid j, k \in \mathbb{N}\}$ be a set of bounded linear transformations $X \to Y$. Suppose for each k, there exists $x \in X$ so that $\sup_j \left\|T_{j,k}x\right\|_Y = \infty$. Then there is an $x \in X$ so that $\sup_j \left\|T_{j,k}x\right\|_Y = \infty$.

Problem 93 : Wisconsin 8/2014 #9

Fix some $n \in \mathbb{N}$. Prove that there are two positive finite measures μ_1 and μ_2 defined on $[0,1]$ so that

$$P'(1) = \int_0^1 P(x)\,d\mu_2(x) - \int_0^1 P(x)\,d\mu_1(x)$$

for all polynomials $P(x)$ of degree at most n. Is the same statement true for polynomials of arbitrary order?

Problem 94 : Wisconsin 8/2013 #5

Let $E = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - x_2 \in \mathbb{Q}\}$. Is it possible to find two Lebesgue measurable sets $A_1, A_2 \subseteq \mathbb{R}$ such that $|A_1|, |A_2| > 0$ and $A_1 \times A_2 \subseteq E^c$?

Problem 95 : Wisconsin 8/2013 #7R

Suppose X is a Banach space and T_n is a sequence of linear bounded operators that map X to itself. Prove that the following statements are equivalent:

(a) If $x_n \in X$ and the series $\sum_{n=1}^{\infty} x_n$ converges in X then $T_n x_n \to 0$.

(b) $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$.

Problem 96 : Purdue 8/2018 #1
Let Δ denotes the symmetric difference, i.e. $A\Delta B = (A \setminus B) \cup (B \setminus A)$.

(a) Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is measurable and $\{E_j\}$ is a sequence of measurable subsets of the unit ball such that

$$\chi_{E_j} \to f \qquad \text{a.e.}$$

as $j \to \infty$. Prove that there exists a measurable set $E \subseteq \mathbb{R}^d$ such htat

$$m(E\Delta E_j) \to 0 \qquad \text{as } j \to \infty$$

(b) If $E \subseteq \mathbb{R}^d$ is a measurable set such that $m(E) \in [0, \infty)$, show that

$$m(E\Delta(E + t)) \to 0$$

as $|t| \to 0$. Does this hold if $m(E) = \infty$? Justify your answer.

Problem 97 : Purdue 8/2018 #2
Suppose $f : [0, 1] \to \mathbb{R}$ is absolutely continuous with

$$\int_0^1 e^{|f'|}\, dx \leq C$$

for some $C > 0$.

(a) Prove that

$$\sup_{x\neq y\in[0,1]} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} < \infty$$

for all $\alpha \in (0, 1)$.

(b) Given $\alpha \in (0, 1)$ and $y \in (0, 1)$, determine

$$\lim_{x\to y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}}$$

Problem 98 : Purdue 1/2018 #2

(a) Construct a sequence of continuous functions $\{f_n\}$ with $f_n : [0,1] \to [0,1]$ such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0$$

but such that the sequence $\{f_n(x)\}$ converges for no $x \in [0,1]$. Deduce thus that convergence in norm does not imply a.e. convergence.

(b) Show that in the setting described by (a) one can always extract a subsequence $\{f_{n_k}\}$ which converges m-a.e. at $f = 0$.

(c) How about the following partial reverse implication: is it true that if $\{f_n\}$ is a sequence of continuous functions with $f_n : [0,1] \to [0,1]$ and such that

$$\exists \lim_{n\to\infty} f_n(x) = 0 \text{ a.e. } x\ in[0,1]$$

then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0$$

must hold? Justify your answer.

Problem 99 : Purdue 8/2017 #4

Give an example of a sequence of functions $\{f_n\}$ in $L^1(\mathbb{R})$, such that

$$\lim_{n\to\infty} f_n(x) \text{ exists a.e.}$$

and that $f := \lim_{n\to\infty} f_n$ is again in $L^1(\mathbb{R})$, but

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \neq \int_{\mathbb{R}} f$$

Problem 100 : Purdue 1/2017 #1

Let f be a non-negative measurable function on a finite measure space $(X, \mathcal{F}, \mu)$. Prove that the sequence $\int_X f^n\,d\mu$ for $n = 1, 2, \cdots$ tends to either $+\infty$ or to $\mu(\{x \in X \mid f(x) = 1\})$.

Problem 101 : Purdue 1/2017 #2

Consider $[0,1]$ with its Lebesgue measure. Let $1 < p < \infty$ and set

$$\Gamma = \left\{ f \in L^p[0,1] \ :\ \int_0^1 5f(x)\,x^3\,dx \leq \frac{1}{\pi}\int_0^1 f(x)\,dx \right\}$$

(a subset of the metric space $M = L^p[0,1]$ with the metric $d_M(f,g) = \|f-g\|_p$). Prove that Γ is closed in $L^p[0,1]$ (equivalently, its complement is open).

Problem 102 : Purdue 1/2017 #3
Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$. Let $E_1, \cdots, E_{50}$ be measurable sets with the property that almost every $x \in X$ belongs to at least 10 of these sets. Prove that at least one of these sets must have measure greater than or equal to $1/5$.

Problem 103 : Purdue 1/2017 #4
Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. Prove that the following limit exists and find it:

$$\lim_{m\to\infty} \int_0^\infty \frac{e^{-x} f(x+2)}{2^m x^2 + 2^{-m}}\, dx$$

Problem 104 : Purdue 8/2016 #1
Recall that the distance between two disjoint, non-empty sets $S, T \subseteq \mathbb{R}$ is defined as

$$d(S,T) := \inf\{|s-t| \mid s \in S\, t \in T\}$$

Assume that $d(S,T) > 0$. Show that $m^*(S \cup T) = m^*(S) + m^*(T)$. (Here $m^*(\cdot)$ is the outer Lebesgue measure.)

Problem 105 : Purdue 8/2016 #4
Let $\{f_n\}$ be a sequence of functions in $L^2(\mathbb{R})$ and let $f \in L^2(\mathbb{R})$. Assume that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n g\, dx = \int_{\mathbb{R}} f_n g\, dx$$

for all $g \in L^2(\mathbb{R})$.

(a) Show that $\|f\|_2 \le \varliminf_{n\to\infty} \|f_n\|_2$.

(b) Suppose in addition that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n^2\, dx = \int_{\mathbb{R}} f^2\, dx$$

Prove that $\|f_n - f\|_2 = 0$.

(c) Give an example where the inequality in (a) is stricty, with the right-hand side being finite.

Problem 106 : Purdue 1/2016 #5
Consider the measure space $(\Omega, \mathcal{A}, \mu)$ and let $1 < a < b < \infty$ $\psi \in L^a \cap L^b$. Denoting the L^p-norm by $\|\cdot\|_p$, prove that the function

$$p \mapsto \|\psi\|_p \in \mathbb{R}$$

is continuous on $[a, b]$.

Problem 107 : Purdue 8/2015 #1

Prove that the following limit exists and find it:

$$\lim_{n\to\infty} \int_{[0,1]} n \left(1 + \frac{x}{n^2}\right)^{-2} x^{-3/2} \sin\left(\frac{x}{n}\right) dm(x)$$

Problem 108 : Purdue 8/2015 #2

Assume $1 \le p < \infty$, $\{g_k\}$ and g are in $L^p(\mathbb{R}^n)$, and $g_k \to g$ in $L^p(\mathbb{R}^n)$. Prove that if $\{\lambda_k\} \subseteq \mathbb{R}^n$ and $\lambda_k \to 0$, then

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |g_k(x + \lambda_k) - g_k(x)|^p \, dm(x) = 0$$

Problem 109 : Purdue 8/2015 #3

Assume f is a Lebesgue measureable function on $\mathbb{R}$ satisfying

(i) there exists $p \in (1, \infty)$ such that $f \in L^p(I)$ for every bounded interval I in $\mathbb{R}$,

(ii) there exists $\theta \in (0, 1)$ such that

$$\left|\int_I f \, dm\right|^p \le \theta \cdot m(I)^{p-1} \cdot \int_I |f|^p \, dm$$

Prove that $f = 0$ almost everywhere in $\mathbb{R}$.

Problem 110 : Purdue 8/2015 #5

Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Assume $\{f_j\}$ and f are measurable functions from X to $[0, \infty)$ such that $f_j(x) \to f(x)$ pointwise a.e. on X as $j \to \infty$, and for some constant C,

$$\int_X f_j \, d\mu \le C \text{ for all } j$$

Prove that $\{\log(1 + f_j(x))\}$ and $\log(1 + f(x))$ are measurable functions on X and $\log(1 + f_j)$ converges to $\log(1 + f)$ in $L^1(X)$.

Problem 111 : Purdue 1/2015 #3

Suppose that $1 < p < \infty$. We say that a sequence $\{f_n\}$ in $L^p[0, 1]$ converges weakly to $f \in L^p[0, 1]$ if $\phi(f) \to \phi(f)$ for every bounded linear functional ϕ on $L^p[0, 1]$. Assume that $\|f_n\|_p \le 1$ and that $f_n \to 0$ almost everywhere. Prove that f_n converges weakly to 0. (Hint: use Egoroff theorem.)

Problem 112 : Purdue 1/2015 #4
Suppose that $A, B \subseteq [0, 1]$ are measurable sets each of Lebesgue measure at least 1/2. Prove that there exists $x \in [-1, 1]$ such that the measure of $(A + x) \cap B$ is at least $1/10$.

Problem 113 : Purdue 1/2015 #5
Suppose that $p > 4/3$ and that $f \in L^p(\mathbb{R})$. Prove that

$$\lim_{t \to 0^+} \int_0^t x^{-1/4} f(x)\, dx = 0$$

Problem 114 : Purdue 8/2014 #6
Let $f, g \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Show that

$$\psi(x) = \int_{\mathbb{R}^n} f(x + y)\, g(y)\, dy$$

is an uniformly continuous function in $\mathbb{R}^n$.

Problem 115 : Purdue 1/2014 #4
Let $(X, \mathcal{M}, \mu)$ be a measure space, T a metric space, and $f : X \times T \to \mathbb{R}$ a function. Assume that $f(\cdot, t)$ is a measurable function for each $t \in T$ and $f(x, \cdot)$ is a continuous function for each $x \in X$. Assume also that there exists an integrable function g such that for each $t \in T$ we have $|f(x, t)| \le g(x)$ for almost all $x \in X$. Show that the function $F : T \to \mathbb{R}$, defined by

$$F(t) = \int_X f(x, t)\, d\mu(x)$$

is a continuous function.

Problem 116 : Ohio State 6211 8/2019 #3
Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let f and g be real-valued integrable functions such that $\int_X f\, d\mu = \int_X g\, d\mu$. Prove that either (i) $f = g$ a.e., or (ii) there exists $E \in \mathcal{M}$ such that $\int_E f\, d\mu > \int_E g\, d\mu$.

Problem 117 : Ohio State 6211 2019 #5
Let $(X, \mathcal{M}, \mu)$ be a measure space and f an integrable function on X. Show that given $\varepsilon > 0$, there exists $\delta > 0$ such that for every measurable set $E \in \mathcal{M}$ with $\mu(E) < \delta$, we have $\int_E |f| < \varepsilon$.

Problem 118 : Ohio State 6211 2018 #1
Let $(X, \mathcal{A}, \mu)$ be a semifinite measure space and let $f, g : X \to [0, \infty]$ be measurable. Suppose $\int_A f\, d\mu \le \int_A g\, d\mu$ for each $A \in \mathcal{A}$. Prove that $f \le g$ μ-a.e. (Note: A proof that does not involve subtraction is to be preferred, since we want to avoid subtracting infinity from infinity).

Problem 119 : Ohio State 6212 2018 #5
Let E and G be Banach spaces, let F be a normed linear space, and let $S : E \to F$ and $T : F \to G$ be linear, continuous, one-to-one, and onto. Prove that F is a Banach space.

Problem 120 : Ohio State 6211 2017 #1
Suppose that C is a non-empty collection of open balls in $\mathbb{R}^n$, and let $U = \bigcup_B \in C$. Show that if $c < m(U)$, then there are *disjoint* $B_1, \cdots, B_k \in C$ such that $\sum_{i=1}^k m(B_i) > 3^{-n}c$. (Note: you may not use the Vitali Covering Lemma without proof).

Problem 121 : Ohio State 6211 2016 #3

(a) State the Monotone Convergence Theorem.

(b) State Fatou's Lemma.

(c) Assuming the Monotone Convergence Theorem, prove Fatou's Lemma.

(d) Assuming Fatou's Lemma, prove Monotone Convergence Theorem.

Problem 122 : Ohio State 6211 2015 #4
Let $(Y, \mathcal{B}, \mu)$ be a measurable space. Let $f : Y \to \mathbb{C}$ and $g : Y \to \mathbb{R}$ be measurable functions. Suppose $\int_Y |f| \, d\mu < \infty$. Define $h : \mathbb{R} \to \mathbb{C}$ by $h(y) = \int_Y f(y) e^{i x g(y)} \, d\mu(y)$. Suppose also that $\int_Y |fg| \, d\mu < \infty$. Prove that h is differentiable on $\mathbb{R}$ and find its derivative h'. You may use the DCT but not the differentiation under integral sign without proof.

Problem 123 : Ohio State 6212 2015 #1
Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f : X \to [0, \infty]$ be measurable. Define $\nu : \mathcal{A} \to [0, \infty]$ by $\nu(A) = \int_A f \, d\mu$. Then ν is a measure on $\mathcal{A}$. Suppose in addition that $\mu(X) < \infty$.

(a) Prove that ν is expressible in the form $\nu = \sum_n \nu_n$, where $\{\nu_n\}$ is a sequence of finite measures on $\mathcal{A}$.

(b) Prove that ν is σ-finite iff $f < \infty$ μ-a.e.

Problem 124 : Ohio State 6212 2015 #2
Let $(X, \mathcal{A}, \nu)$ be a mesaure space with $\nu(X) < \infty$.

(a) Let $\mathcal{E}$ be a subset of $\mathcal{A}$ such that $\mathcal{E}$ is closed under countable unions. Prove that $\mathcal{E}$ has a ν-essentially largest element, i.e. there is $E \in \mathcal{E}$ so that for each $D \in \mathcal{E}$, $\nu(D \setminus E) = 0$.

(b) Let μ be any measure on $\mathcal{A}$. Prove that ν has a Lebesgue decomposition with respect to μ.

Problem 125 : Texas A&M 8/2019 #4
Let E be a subset $\mathbb{R}$ which is not Lebesgue measurable. Prove that there exists an $\eta > 0$ so that for any two Lebesgue measurable sets A, B satisfying $A \subset E \subset B$ one has $\lambda(B \setminus A) > \eta$.

Problem 126 : Texas A&M 8/2018 #2

Let μ be a positive measure. Suppose that $\{f_n\}$ is a Cauchy sequence in $L^1(\mu)$. Show that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies

$$\left| \int_E f_n \, d\mu \right| < \varepsilon \qquad \forall n \geq 1$$

(Hint: it may be helpful to first show the following: suppose μ and ν are positive measures on the same measurable space with ν finite and absolutely continuous with respect to μ. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $\nu(E) < \varepsilon$.)

Problem 127 : Texas A&M 1/2018 #5

Prove that the following limit exists and compute its value:

$$\lim_{n\to\infty} \int_0^n \left(\sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} \right) e^{-2x} \, dx$$

Problem 128 : Texas A&M 1/2018 #10

SUppose $(X, \mathcal{M}, \rho)$ is a finite measure space and $\mathcal{A} \subseteq \mathcal{M}$ is an algebra of sets with a finitely additive complex measure $\mu : \mathcal{A} \to \mathbb{C}$ such that $|\mu(E)| \leq \rho(E)$ for all $E \in \mathcal{A}$. Show that there exists a complesx measure $\nu : \mathcal{M} \to \mathbb{C}$ whose restriction to $\mathcal{A}$ is μ and such that $|\nu(E)| \leq \rho(E)$ for all $E \in \mathcal{M}$. (Hint: you may want to consider the subspace $V \subseteq L^1(\rho)$ that is spanned by the set of characteristic functions χ_E for $E \in \mathcal{A}$, and a certain linear functional on V.)

Problem 129 : Texas A&M 8/2017 #8

Let $\{f_n\}$ be a sequence of continuous functions on $\mathbb{R}$ that converges pointwise to a real-valued function f. Prove that for each $a < b$, the functions f is continuous at some point of $[a, b]$. (Hint: let $E_{n,m,k} = \{|f_n - f_m| \leq 1/k\}$.)

Problem 130 : Texas A&M 8/2016 #10

If A is a Borel subset of the line, then $E = \{(x, y) \mid x - y \in A\}$ is a Borel subset of the plane. If the Lebesgue measure of A is 0, show that the Lebesgue measure of E is 0.

Problem 131 : Texas A&M 1/2016 #6

Let $\{f_k\}$ be a sequence of increasing functions on $[0, 1]$. Suppose

$$\sum_{k=1}^{\infty} f_k(x)$$

converges for all $x \in [0, 1]$. Denote the limit function by f, that is

$$f(x) := \sum_{k=1}^{\infty} f_k(x)$$

Prove that

$$f'(x) = \sum_{k=1}^{\infty} f_k'(x) \qquad \text{a.e. } x \in [0,1]$$

Problem 132 : Michigan 9/2019 #1

Let E be the set of all $x \in (0,1)$ such that there exists a sequence of irreducible fractions $\{p_n/q_n\}$ with $p_n, q_n \in \mathbb{N}$, $q_1 < q_2 < \cdots$ such that

$$\left| x - \frac{p_n}{q_n} \right| \le \frac{1}{q_n^3}$$

for all n. Prove that the Lebesgue measure of E is zero.

Problem 133 : Michigan 1/2019 #1

Let $f_n : (0,1) \to \mathbb{R}$ be a sequence of measurable functions and consider the sequence $g_n : \mathbb{R} \to \mathbb{R}$ of functions defined by

$$g_n(x) = \frac{f_1(x) + \cdots + f_n(x)}{n}$$

(a) Suppose that there is a function $f : (0,1) \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost all $x \in (0,1)$. Show that $\lim_{n\to\infty} g_n(x) = f(x)$ for almost all $x \in (0,1)$.

(b) Let $f_n(x) = \sin(n\pi x)$, $0 < x < 1$. Show that $\lim_{n\to\infty} g_n(x) = 0$ for almost all $x \in (0,1)$, but also that for alost all $x \in (0,1)$ the limit $\lim_{n\to\infty} f_n(x)$ fails to exist.

Problem 134 : Michigan 1/2019 #2

Let p satisfies $1 < p < \infty$ and $f : (0,\infty) \to \mathbb{R}$ be in $L^p(0,\infty)$. Prove that

$$\int_0^\infty \left(\int_0^1 |f(sx)|\, ds \right)^p dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty |f(x)|^p\, dx$$

(Hint: you may quote Minkowski integral inequality.)

Problem 135 : Michigan 1/2019 #3

Let $f : [0,1] \to \mathbb{R}$ be defined by $f(0) = 0$ and $f(x) = x^2 \sin(x^{-2})$ for $0 < x \le 1$. Is f of bounded variation?

Problem 136 : Michigan 9/2018 #1

Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function which has the property that

$$m(|f| > \alpha) \le \frac{1}{1+\alpha^3} \qquad \text{for } \alpha > 0$$

(a) Show that $|f|^p$ is integrable for $p < 3$.

(b) Give an example of a function satisfying the given property for which $|f|^3$ is not integrable.

Problem 137 : Michigan 9/2018 #2

Let $H : \mathbb{R} \to \mathbb{R}$ be the periodic function with period 1 which is given in the interval $[0, 1)$ by $H(x) = 1$ if $0 \le x < 1/2$ and $H(x) = -1$ if $1/2 \le x < 1$. Consider the sequence of functions $H_n \in L^2(0, 1)$, $n \ge \mathbb{N}$, defined by $H_n(x) = H(2^n x)$, $0 < x < 1$.

(a) Show that

$$\lim_{n\to\infty} \int_0^1 g(x)H_n(x)\,dx = 0 \text{ for all } g \in L^2(0,1)$$

(b) Show that the sequence $H_n(\cdot)$, $n \in \mathbb{N}$, has no convergence subsequence in $L^2(0, 1)$.

Problem 138 : Michigan 5/2018 #2

Provide a proof or a counter-example to the following statement: If E is a bounded open subset of $\mathbb{R}$ then the boundary of E has Lebesgue measure zero.

Problem 139 : Michigan 5/2018 #3

Show that $\{f \in L^2(\mathbb{R}, m) \mid \int_{\mathbb{R}} |f| = \infty\}$ is dense in $L^2(\mathbb{R}, m)$.

Problem 140 : Michigan 5/2018 #4

Let μ be a non-negative measure on the interval $(-1, 1)$ with the property that all open subintervals of $(-1, 1)$ are μ-measurable and $\mu((-1, 1)) = 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous and let $f_n : \mathbb{R} \to \mathbb{R}$ be the function defined by $f_n(x) = \int_{-1}^{1} f\left(x + \frac{t}{n}\right) d\mu(t)$.

(a) Show that each f_n is uniformly continuous.

(b) Show that the f_n's converge uniformly to f.

Problem 141 : Michigan 5/2018 #5

Let f_n be a sequence of functions in $L^\infty([0, 1], m)$ satisfying the conditions

(i) $\|f_n\|_\infty \le 1$, and

(ii) $\int_a^b f_n\,dm \to 0$ for all $0 \le a < b \le 1$.

(a) Show that $\int_0^1 f_n g\,dm \to 0$ for all $g \in L^1([0, 1], m)$.

(b) Under assumptions (i) and (ii), does $f_n \to 0$ in $L^1([0, 1], m)$?

Problem 142 : Michigan 1/2018 #1

Let $f_j : [0,1] \to [0,1]$ be a sequence of integable functions satisfying $\int f_j \to 0$. Prove or disprove: we must have $f_j \to 0$ almost everywhere.

Problem 143 : Michigan 1/2018 #3

Let $E \subseteq \mathbb{R}$ be a measurable set of positive measure and let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative measurable function with positive integral. Show that there exists $t \in \mathbb{R}$ so that $\int_{E+t} f > 0$.

Problem 144 : Michigan 1/2018 #1

Suppose we are given

1. a bounded continuous function f on $\mathbb{R}$;
2. an integrable function g on $\mathbb{R}$.

For $t > 0$, let $h(t) = \int_{\mathbb{R}} f(tx)\, g(x/t)\, dx$.

(a) Must h be continuous?

(b) Must h be bounded?

(c) Must $\lim_{h \searrow 0} h(t)$ exist?

Problem 145 : Michigan 9/2017 #2

Let $\phi : [-1,1] \to [-1,1]$ be a non-decreasing function with $\phi(-1) = -1$ and $\phi(1) = 1$. Show that

$$\int_{(x,y)\in[-1,1]^2} (\phi(x) - \phi(y))^2\, dA(x,y) \le 8$$

When will equality hold?

Problem 146 : Michigan 9/2017 #4 Let $f \in L^2(I)$, for any finite interval $I \subseteq \mathbb{R}$. Assume that

$$\int_{-a}^{a} |t|\, |f(x+t)|\, dt \ge \frac{2}{\sqrt{3}} a^2$$

for all $a > 0$, $x \in \mathbb{R}$. Show that $|f(x)| \ge 1$ for a.e. $x \in \mathbb{R}$.

Problem 147 : Michigan 5/2017 #1

Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of measurable functions on a measure space $(X, \mathcal{M}, \mu)$. Suppose that the series

$$\sum_{j=1}^{\infty} \mu\{x \in X \mid |f_j(x)| \ge \varepsilon\}$$

converges for every $\varepsilon > 0$. Prove that $f_j(x) \to 0$ almost everywhere on X.

Problem 148 : Michigan 5/2017 #2

Let $E \subseteq [0,1]$ be the middle-third Cantor set. Find a function $f \in C^\infty(\mathbb{R})$ such that $f \geq 0$ and $\{x \in \mathbb{R} \mid f(x) = 0\} = E$.

Problem 149 : Michigan 5/2017 #3

Let $\alpha < 1$. Prove the existence of the limit

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x^{1/n} e^{\alpha x}\, dx$$

and calculate it.

Problem 150 : Michigan 5/2017 #5

Construct a function $f \in L^1(\mathbb{R}^n)$ such that $f \notin L^p(U)$ for any open subset $U \subseteq \mathbb{R}^n$ and any $p > 1$.

Problem 151 : Michigan 1/2017 #3

Let R be the unit square $[0,1] \times [0,1]$ in the plane. Let N be the function that assigns to each real number x in the unit interval the positive integer that indicates the first place in the decimal expansion of x after the decimal point where the first 0 occurs. If there are two expansions, use the expansion that ends in a string of zeroes. If 0 does not occur, let $N(x) = \infty$. For example, $N(0.0) = 1$, $N(0.5) = 2$, $N(1/9) = \infty$, and $N(0.4763014\cdots) = 5$. Evaluate $\iint_R y^{-N(x)}\, d\mu$.

Problem 152 : Michigan 9/2016 #1

Construct an open set $U \subseteq [0,1]$ such that

(a) U is dense in $[0,1]$;

(b) U has Lebesgue measure $\mu(U) < 1$;

(c) $\mu(U \cap I) > 0$ for every open interval $I \subseteq [0,1]$.

Problem 153 : Michigan 9/2016 #2

Let $A \subseteq \mathbb{R}$ be a measurable set. Suppose

$$\frac{\mu(A \cap I)}{\mu(I)} \leq \frac{1}{2}$$

for every finite interval $I \subseteq \mathbb{R}$, where μ is the Lebesgue measure on $\mathbb{R}$. Show that $\mu(A) = 0$.

Problem 154 : Michigan 9/2016 #5

Let $1 < p < \infty$ and $f \in L^p[0,\infty)$. Show that

$$\left| \int_0^x f(t)\, dt \right| \leq \|f\|_p\, x^{1-\frac{1}{p}}$$

for every $x > 0$.

Problem 155 : Michigan 5/2016 #1

(a) Prove that for every Borel measurable set $E \subseteq [0,1]$,

$$\lim_{n\to\infty} \int_E \sin(2\pi n x)\,dx = 0$$

(b) Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions satisfying

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0$$

Does it imply that $\lim_{n\to\infty} f_n(x) = 0$? Prove or give a counter-example.

Problem 156 : Michigan 5/2016 #2

Let $f \in L^1[0,1]$ and let g be a bounded increasing function on $[0,1]$. Assume that for any interval $[a,b] \subseteq [0,1]$,

$$\left| \int_a^b f(x)\,dx \right|^2 \leq (g(b) - g(a))(b-a)$$

Prove that $f \in L^2[0,1]$.

Problem 157 : Michigan 5/2016 #3

Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space. Prove that there is a finite measure ν on $(X, \mathcal{B})$ with the property that

$$\nu(E) = 0 \text{ if and only if } \mu(E) = 0$$

Problem 158 : Michigan 5/2016 #4

Assume that $f \in L^1[0,\infty) \cap C[0,\infty)$. Evaluate the limit

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^\infty f(x)\, e^{-x/\varepsilon}\,dx$$

Problem 159 : Michigan 5/2016 #5

Let f be a function on $\mathbb{R}^n$. Assume that for any $\varepsilon > 0$, there exist measurable functions $g, h \in L^1(\mathbb{R}^n)$ such that $g(x) \leq f(x) \leq h(x)$ for all $x \in \mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} (h(x) - g(x))\,dx < \varepsilon$$

Prove that f is measurable on $\mathbb{R}^n$ and $f \in L^1(\mathbb{R}^n)$.

Problem 160 : Michigan 1/2016 #1

Let E be the subset of the interval $[0, 1]$ consisting of the points x that has a decimal expansion

$$x = 0.a_1a_2a_3a_4\cdots$$

with $a_n \neq 5$ for all n. For example, both 0.5 and 0.6 are in E since 0.5 has an expansion $0.5 = 0.4999\cdots$ and $0.6 = 0.600\cdots$. Show that E is Lebesgue measurabe and evaluate the Lebesgue measure of E.

Problem 161 : Michigan 1/2016 #4

Let $f \in L^1[a, b]$. Prove that if

$$\lim_{h\to 0} \frac{1}{h} \int_a^b |f(x+h) - f(x)|\, dx = 0$$

then there is a constant c such that $f(x) = c$ for almost every $x \in (a, b)$.

Problem 162 : Michigan 1/2016 #5

Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported C^1 function. Show that there is a constant $C > 0$, independent of f, such that for all $x \in \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{(f(x) - f(y))^4}{(x-y)^4}\, dy \leq C \left\|f'\right\|_4^4$$

(Hint: you may like to use integration by parts.)

Problem 163 : Michigan 9/2015 #1

Fix $1 < p < \infty$. Let $f \in L^p(E)$, where E is a measurable subset of $\mathbb{R}^d$. Assume that

$$\int_E f(x)\, g(x)\, dx = 0$$

for all compactly supported continuous $g : \mathbb{R}^d \to \mathbb{R}$. Is $f(x) = 0$ for amost every x in E? Prove or counter-example.

Problem 164 : Michigan 9/2015 #2

Let (a, b) be an interval on $\mathbb{R}$. Let $f \in L^1(a, b)$. Assume that

$$\int_a^b f(x)\, g'(x)\, dx = 0$$

for all C^1 functions g with support compactly contained in (a, b). Prove that there is a constant c such that $f(x) = c$ for almost every x in (a, b).

Problem 165 : Michigan 9/2015 #4

Let $f_1(s, t)$, $f_2(s, t)$, and $f_3(s, t)$ be non-negative measurable functions on $\mathbb{R}^2$. Set

$$I_k = \int_{\mathbb{R}^2} (f_k(s,t))^2\, ds\, dt \qquad k = 1, 2, 3$$

Prove that

$$\int_{\mathbb{R}^3} f_1(x_2, x_3)\, f_2(x_1, x_3)\, f_3(x_1, x_2)\, dx_1\, dx_2\, dx_3 \leq (I_1 I_2 I_3)^{1/2}$$

Problem 166 : Michigan 9/2015 #5

Let E be a measurable subset of $\mathbb{R}$ such that $m(E) < \infty$. Let $f \in L^\infty(E)$ with $\|f\|_\infty > 0$. Show that

$$\lim_{n\to\infty} \frac{\|f\|_{n+1}^{n+1}}{\|f\|_n^n} = \|f\|_\infty$$

Problem 167 : Michigan 5/2015 #1

Let $E \subseteq \mathbb{R}$. Show that the characteristic function $\chi_E(x)$ is the limit of a sequence of continuous functions iff E is both F_σ and G_δ.

Problem 168 : Michigan 5/2015 #3

Let $f_k(x)$, $k \in \mathbb{N}$, be increasing functions on $[a, b]$. Assume $\sum_{k=1}^{\infty} f_k(x)$ is convergent on $[a, b]$. Show that

$$\left(\sum_{k=1}^{\infty} f_k(x)\right)' = \sum_{k=1}^{\infty} f_k'(x) \qquad \text{a.e. } x \in [a, b]$$

Problem 169 : Michigan 5/2015 #4

(a) Aassume that $f \in L^\infty(\mathbb{R})$ and f is continuous at 0. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{n}{\pi(1 + (nx))^2} f(x)\, dx = f(0)$$

(b) Assume that $f \in L^\infty(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{n}{\pi(1 + n^2(x - y)^2)} f(y)\, dy = f(x) \qquad \text{a.e. } x \in \mathbb{R}$$

$\left(\text{Hint: } \int_{\mathbb{R}} [\pi(1 + y^2)]^{-1}\, dy = 1\right)$.

Problem 170 : Michigan 5/2019 #5

Let $\{f_n\}$ be a sequence of functions in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, which converges a.e. to a function $f \in L^p(\mathbb{R}^n)$, and suppose that there is a constant M such that $\left\|f_n\right\|_p \leq M$ for all n. Show that for every $g \in L^q(\mathbb{R}^n)$, q the conjugate of p,

$$\int_{\mathbb{R}} fg = \lim_{n\to\infty} \int_{\mathbb{R}} f_n g$$

Is the statement true for $p = 1$? (Hint: you may want to use Egoroff's Theorem.)

Problem 171 : Michigan 1/2015 #1

Let f be a non-negative measurable function on $(0, 1)$. Assume that there is a constant c such that

$$\int_0^1 (f(x))^n \, dx = c \qquad n = 1, 2, \cdots$$

Show that there is a measurable set $E \subseteq (0, 1)$ such that

$$f(x) = \chi_E(x) \qquad \text{for a.e. } x \in (0, 1)$$

Problem 172 : Michigan 1/2015 #2

Let f be locally integrable on $\mathbb{R}^n$, $1 < p < \infty$. Show that the following are equivalent:

1. $f \in L^p(\mathbb{R}^n)$.
2. there exists $M > 0$, such that for any finite collection of mutually disjoint measurable sets $E_1, E_2, \cdots, E_k$, with $0 < m(E_i) < \infty$ for $1 \leq i \leq k$,

$$\sum_{i=1}^{k} \left(\frac{1}{m(E_i)} \right)^{p-1} \left| \int_{E_i} f(x) \, dx \right|^p \leq M$$

Problem 173 : Michigan 1/2015 #3

Let $f : \mathbb{R} \times [0, 1] \to \mathbb{R}$ be a measurable function such that for any $y \in [0, 1]$,

$$\int_{\mathbb{R}} f^2(x, y) \, dx \leq 1$$

Prove that there exists a sequence $x_n \to +\infty$ such that

$$\int_0^1 f(x_n, y) \, dy \to 0$$

Problem 174 : Michigan 1/2015 #4

Let $E_k \subseteq [a, b]$, $k \in \mathbb{N}$, be measurable sets, and there exists $\delta > 0$ such that $m(E_k) \geq \delta$ for all k. Assume that $a_k \in \mathbb{R}$ satisfies

$$\sum_{k=1}^{\infty} |a_k| \chi_{E_k}(x) < \infty \qquad \text{for a.e. } x \in [a, b]$$

Show that

$$\sum_{k=1}^{\infty} |a_k| < \infty$$

(Hint: one of the possible approach uses Egoroff's Theorem.)

Problem 175 : Michigan 1/2015 #5

Let $A, B \subseteq \mathbb{R}^d$. Assume $A \cup B$ is measurable, and $m(A \cup B) < \infty$. If

$$m(A \cup B) = m^*(A) + m^*(B)$$

Show that A and B are measurable. Here $m^*(\cdot)$ is the Lebesgue outer measure. (Hint: prove first that for any set A, there measurable set U, with $A \subseteq U$, such that $m^*(A) = m(U)$.)

Problem 176 : Stein Ch.1 #1

Prove that the middle-third Cantor set C is totally disconnected and perfect. In other words, given two distinct points $x, y \in C$, there is a point $z \notin C$ that lies between x and y, and yet C has no isolated points.

Problem 177 : Stein Ch. 1#5

Suppose E is a given set and $\mathcal{O}_n$ is the open set

$$\mathcal{O}_n = \{x \mid d(x, E) < 1/n\}$$

Show

(a) If E is compact, then $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$.

(b) However the conclusion in (a) may be false for E closed and unbounded, or E open and bounded.

Problem 178 : Stein Ch. 1 #13

The following deals with G_δ and F_σ sets:

(a) Show that a closed set is a G_δ and an open set an F_σ. (Hint: If F is closed, consider $\mathcal{O}_n = \{x \mid d(x, F) < 1/n\}$.)

(b) Give an example of an F_σ which is not a G_δ. (Hint: This is more difficult; let F be a countable set that is that is dense)

(c) Give an example of a Borel set which is neither a G_δ nor an F_σ.

Problem 179 : Stein Ch. 1 #16

(**Borel-Cantelli Lemma**) Suppose $\{E_k\}_{k=1}^\infty$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty$$

Let $E = \limsup_{k\to\infty} E_k$.

(a) Show that E is measurable.

(b) Prove $m(E) = 0$.

Problem 180 : Stein Ch. 1 #17
Let $\{f_n\}$ be a sequence of measurable functions on $[0, 1]$ with $|f_n(x)| < \infty$ for a.e. x. Show that there exists a sequence c_n of positive real numbers such that

$$\frac{f_n(x)}{c_n} \to 0 \qquad texta.e.x$$

(Hint: Pick c_n such that $m(\{x \mid |f_n(x)/c_n| > 1/n\}) < 2^{-n}$, and apply the Borel-Cantelli lemma.)

Problem 181 : Stein Ch. 1 #30
If E and F are measurable, and $m(E), m(F) > 0$, prove that

$$E + F = \{x + y \mid x \in E,\ y \in F\}$$

contains an interval. (Hint: use Steinhaus Theorem)

Problem 182 : Stein Prop. 2.5
Suppose $f \in L^1(\mathbb{R}^d)$. Then

$$\|f_h - f\|_1 \to 0 \qquad \text{as } h \to 0$$

Problem 183 : Stein Ch. 2 #2
In analogy to Prop. 2.5, prove that if f is integrable on $\mathbb{R}^d$ and $\delta > 0$, then $f(\delta x)$ converges to $f(x)$ i in the L^1-norm as $\delta \to 1$.

Problem 184 : Stein Ch. 2 #4
Suppose f is integrable on $[0, b]$ and

$$g(x) = \int_x^b \frac{f(t)}{t}\,dt \qquad \text{for } 0 < x \le b$$

Prove that g is integrable on $[0, b]$ and

$$\int_0^b g(x)\,dx = \int_0^b f(t)\,dt$$

Problem 185 : Stein Ch. 2 #5
Suppose F is a closed set in $\mathbb{R}$, whose complement has finite measure, and let $\delta(x)$ denots the distance form x to F. Consider

$$I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x - y|^2}\,dy$$

(a) Prove that δ is continuous by showing it is Lipschitz.

(b) Show that $I(x) = \infty$ for each $x \notin F$.

(c) Show that $I(x) < \infty$ for a.e. $x \in F$. (Hint: consider $\int_F I(x)\,dx$.)

Problem 186 : Stein Ch. 2 #10 Suppose $f \geq 0$, and let $E_{2^k} = \{f(x) > 2^k\}$ and $F_k = \{2^k < f(x) \leq 2^{k+1}\}$. If f is finite a.e., then

$$\bigcup_{k=-\infty}^{\infty} F_k = \{f(x) > 0\}$$

and the sets F_k are disjoint.
Prove that f is integrable if and only if

$$\sum_{k=-\infty}^{\infty} 2^k\, m(F_k) < \infty \qquad \text{if and only if} \qquad \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty$$

Use this result to verifiy the following assertions. Let

$$f(x) = \begin{cases} |x|^{-a} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} |x|^{-b} & \text{if } |x| > 1 \\ 0 & \text{otherwise} \end{cases}$$

Then f is integrable on $\mathbb{R}^d$ iff $a < d$; also g is integrable on $\mathbb{R}^d$ iff $b > d$.a

Problem 187 : Stein Ch. 2 #13
Give an example of two measurable sets A and B such that $A + B$ is not measurable. (Hint: in $\mathbb{R}^2$ take $A = \{0\} \times [0,1]$ and $B = \mathcal{N} \times \{0\}$, where $\mathcal{N}$ is the non-measurable set in $[0,1]$.)

Problem 188 : Stein Ch. 2 #18
Let f be a measurable finite-valued function on $[0,1]$ and suppose that $|f(x) - f(y)|$ is integrable on $[0,1] \times [0,1]$. Show that $f(x)$ is integrable on $[0,1]$.

Problem 189 : Stein Ch. 2 #21 (a)-(d)
Suppose that f and g are measurable functions on $\mathbb{R}^d$.

(a) Prove that $f(x-y)g(y)$ is measurable on $\mathbb{R}^{2d}$.

(b) Show that if f and g are integrable on $\mathbb{R}^d$, then $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2d}$.

(c) Define the convolution of f and g by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy$$

Show that $f * g$ is well-defined for a.e. x (that is, $f(x-y)g(y)$ is integrable on $\mathbb{R}^d$ for a.e. x).

(d) Show that $f * g$ is integrable whenever f and g are integrable, and that

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

with equality if f and g are non-negative.

Problem 190 : Washington FA19 #2
Let H_1 and H_2 be Hilbert space with scalars in $\mathbb{R}$. Let $T : H_1 \to H_2$ be a linear isometry from H_1 into (but not necessarily onto) H_2, i.e. $\|Tx\| = \|x\|$ for all $x \in H_1$. Prove that $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in H_1$.

Problem 191 : Washington FA19 #8 Let $1 \leq p < \infty$. For every $f \in L^p[0,1]$ and every integer $n \geq 1$, define the piecewise constant function f_n by

$$f_n(x) = \begin{cases} n \int_{k/n}^{(k+1)/n} f(x)\, dx & \text{for } \frac{k}{n} \leq x < \frac{k+1}{n} \text{ when } 0 \leq k \leq n-2 \\ n \int_{k/n}^{(k+1)/n} f(x)\, dx & \text{for } \frac{k}{n} \leq x \leq \frac{k+1}{n} \text{ when } k = n-1 \end{cases}$$

1. Show that for every $f \in L^p[0,1]$, $\left\|f_n\right\|_p \leq \|f\|_p$.
2. Show that for every continuous f, f_n converges uniformly to f.
3. Use parts (a) and (b) and the density of continuous functions in $L^p[0,1]$ to show that, for any $f \in L^p[0,1]$, the sequence $\{f_n\}$ converges to f in $L^p[0,1]$.

Problem 192 : Washington SP19 #2
Let $\{f_n\}$ be a sequence of continuously differentiable functions on $[0,1]$, and assume that

$$|f_n'(x)| \leq \frac{1}{\sqrt{x}} \text{ for all } 0 < x \leq 1 \text{ and all } n \geq 1$$

and that

$$\int_0^1 f_n(x)\, dx = 0 \text{ for all } n \geq 1$$

Prove that this sequence has a subsequence that converges uniformly on $[0,1]$.

Problem 193 : Washington SP19
Suppose $f \in L^1[0,1]$. Show that if $\int_0^1 f(x)(\sin x)^n\, dx = 0$ for every $n \in \mathbb{N}$, then $f = 0$ a.e. on $[0,1]$.

Problem 194 : Washington FA18 #1
Let μ be a regular Borel measure on $\mathbb{R}^n$ such that $\mu(B) < \infty$ for any ball $B \subseteq \mathbb{R}^n$. Let E be a Borel set with $\mu(E) > 0$ such that

$$\overline{\lim_{r\to 0}} \frac{\mu(B_{2r}(x))}{\mu(B_r(x))} < \infty \text{ for all } x \in E$$

Show that there exist a closed set $E_0 \subseteq E$ with $0 < \mu(E_0) < \infty$ and constants $r_0 > 0$ and $M \geq 1$ such that for all $x \in E_0$ and $0 < r < r_0$

$$\frac{\mu(B_{2r}(x))}{\mu(B_r(x))} \leq M$$

Problem 195 : Washington FA18 #2

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with Lipschitz constant L. Show that for every Borel set E,

$$m(f(E)) \leq L^n\, m(E)$$

Problem 196 : Washington FA18 #4

Let $(E, \mathcal{F}, \mu)$ be a probability measure space, that is, μ is a positive measure and $\mu(E) = 1$. Suppose that $\{X_n\}$ is a sequence of measurable functions on this space that is uniformly integrable; that is, for every $\varepsilon > 0$, there is some $M > 0$ so that

$$\int_{\{|X_n|>M\}} |X_n|\, d\mu < \varepsilon \qquad \text{for every } n \geq 1$$

Show that $\lim_{n\to\infty} \frac{1}{n} \int_E \sup_{1\leq k\leq n} |X_k|\, d\mu = 0$.

Problem 197 : Washington 2017 #1

For $k = 0, 1, 2, \cdots$, prove the following identities involving Lebesgue integrals:

(a) The gamma function identity:

$$\int_0^\infty y^k e^{-y}\, dy = k!$$

(b) The incomplete gamma function identity for any $t > 0$:

$$\frac{1}{k!} \int_t^\infty y^k e^{-y}\, dy = e^{-t} \sum_{j=0}^{k} \frac{t^j}{j!}$$

Problem 198 : Washington 2017 #8

Let $a \in \mathbb{R}$ and $b \geq 0$ be constants. Let f be a non-negative Borel measurable function that is bounded on bounded intervals and satisfies

$$0 \leq f(t) \leq a + b \int_0^t f(s)\, ds \qquad \text{for all } t \geq 0$$

Use recursion to prove that $f(t) \leq ae^{bt}$ for all $t \geq 0$.

Problem 199 : Washington 2016 #2

Suppose that a topological space X is compact and $f : X \to \mathbb{R}$ is a (not necessarily continuous) function. If for every $t \in \mathbb{R}$, $f^{-1}([t, \infty))$ is closed, then there is some $x_0 \in X$ so that $f(x_0) = \sup_{x \in X} f(x) < \infty$.

Problem 200 : Washington 2016 #5

If a closed subset K of $\mathbb{R}$ is the union of half-open, half-closed intervals $\{[a_\lambda, b_\lambda) \mid \lambda \in \Lambda\}$, then it is the union of countably many of these intervals.

Problem 201 : Washington 2016 #7

Suppose $f \in L^1[0, 1]$ and g is a bounded Lebesgue measurable periodic function on $\mathbb{R}$ with period 1 (i.e. $g(x + 1) = g(x)$ for all x). Show that

$$\lim_{n\to\infty} \int_0^1 f(x)\, g(nx)\, dx = \int_0^1 f(x)\, dx \cdot \int_0^1 g(x)\, dx$$